U.G. 5th Semester Examination 2021

MATHEMATICS (Honours)

Paper : DSE-1

(CBCS)

Full Marks : 32 Time : 2 Hours

The figures in the margin indicate full marks.
Notations and symbols have their usual meanings.

DSE-1A

Advanced Algebra

Group - A

(4 Marks)

1. Answer any ***four*** questions : 4×1=4

(a) Find the order of the element $(\overline{6}, \overline{4})$ in the group $\mathbb{Z}_{24} \oplus \mathbb{Z}_{16}$.

(b) Give an example of an infinite noncommutative group.

(c) Prove that every group of order 15 is cyclic.

(d) If G is a finite group with exactly two conjugacy classes, then show that $|G| = 2$.

(e) Let $(G, 0)$ be a group and $\rho : G \times G \to G$ be a mapping defined by $\rho(g, x) = x_0 g^{-1}, x, g \in G$. Show that ρ is a group action on G.

(f) Find a prime element in $\mathbb{Z}_{10}$ which is not irreducible.

(g) Show that the polynomial $2x^5 + 10x^3 + 10x + 5$ is irreducible over $\mathbb{Z}$.

Group - B

(10 Marks)

Answer any ***two*** questions : $2\times5=10$

2. (a) Let G be a cyclic group of order mn, whre m, n are positive integers such that $gcd(m, n) = 1$. Show that $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$. 3

   (b) Let G be a noncommutative group of order p^3, where p is a prime. Show that $|Z(G)| = p$. 2

3. Show that no group of order 56 is simple. 5

4. Let G be a group of order 63. If G contains a unique subgroup of order 9 in G, then prove that G is a commutative group. 5

5. Prove that $\mathbb{Z}\left[\sqrt{3}\right] = \left\{a + b\sqrt{3} \mid a,\, b \in \mathbb{Z}\right\}$ is a Euclidean domain. 5

Group - C

(18 Marks)

Answer any ***two*** questions : $2\times9=18$

6. (a) Find the number of elements of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$. 4

   (b) If R is an integral domain, then show that $R[x]$ is also an integral domain. 5

7. (a) Let G be a group of order 30. Show that G is not simple. 5

   (b) If K is a field, prove that $K\left[x\right]$ is a Euclidean domain. 4

8. (a) Show that in the integral domain $\mathbb{Z}\left[i\sqrt{5}\right], 2 + i\sqrt{5}$ is an irreducible element, but not a prime element. 4

   (b) Find all irreducible polynomials of degree 2 over the field $\mathbb{Z}_3$. 5

DSE-1B
[Number Theory]

Full Marks : 32 Time : 2 Hours

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Group - A
(4 Marks)

1. Answer any ***four*** questions : 4×1=4

   (a) Give an example to show that the following conjecture is not always true : Every positive integer can be written in the form $p+a^2$, where p is either a prime or 1 and $a \geq 0$.

   (b) State the Fermat's little theorem.

   (c) Find the number of positive divisors of the number 180.

   (d) Define the Euler's phi-function $\phi(n), n \geq 1$. Compute $\phi(10)$.

   (e) Determine the order of the integer 5 modulo 23.

   (f) Find the least primitive root of the integer 83.

   (g) Encrypt the plaintext message "RETURN HOME" using Caesar cipher.

Group - B

(10 Marks)

Answer any ***two*** questions : 5×2=10

2. Let p be a prime and a be an integer such that $p \nmid a$. Prove that $a^{p-1} \equiv 1 \pmod{p}$. 5

3. Let F and f be two number-theoretic functions related by the formula $F(n)=\sum_{d|n} f(d)$.

   Prove that $f(n)=\sum_{d\,|\,n} \mu(d) F\left(\frac{n}{d}\right)=\mu\left(\frac{n}{d}\right)F(d)$, where μ is the Möbius μ-function. 5

4. (i) Find the remainder when 777^{777} is divided by 16.

(ii) If p and p+2 be a pair of twin prime, prove that

$4(p-1)!+p+4 \equiv 0(\bmod(p+2))$. 3+2

5. Encrypt the message "GOOD CHOICE" using an exponential cipher with modulus $p = 2609$ and exponent $k = 7$. 5

Group - C

(18 Marks)

Answer any ***two*** questions : 9×2=18

6. (a) If p_n is the nth prime number, show that $p_n \leq 2^{2^{n-1}}$. 3

(b) Find the remainder when 1! + 2! + 3! + 4! +......+ 99! + 100! is divided by 12. 3

(c) If n is an odd pseudoprime, show that $2^n - 1$ is a larger one. 3

7. (a) If n and r are positive integers with $1 \leq r < n$, prove that the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is also an integer. 4

(b) If p is a prime and $k > 0$, show that $\phi(p^k) = p^k\left(1-\frac{1}{p}\right)$. 3

(c) Apply Euler's theorem to establish : for any integer $a, a^{37} \equiv a(\bmod 1729)$. 2

8. (a) Let an integer a have order k modulo n. Prove that a^h has order k modulo n if and only if $\gcd(h, k) = 1$. 4

(b) Given that 3 is a primitive root of 43, determine all the positive integers less than 43 having order 6 modulo 43. 3

(c) Find the value of the Legendre symbol (19/23). 2

DSE-1C
[Bio Mathematics]

Full Marks : 32 Time : 2 Hours

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Group - A

(4 Marks)

1. Answer any ***four*** questions : 4×1=4

(a) What is Allee Effect?

(b) What is an epidemic?

(c) Define Critical point.

(d) What do you mean by open access fishery?

(e) Find the equilibrium point of the difference equation $X_{t+1} = rX_t\left(1 - X_t\right)$.

(f) Define carrying capacity.

(g) In discrete model $N_{t+1} = \left(1+r\right)N_t - \frac{r}{k}N_t^2$, $r > 0$, $k > 0$ (constant), for which value of r the non-trivial equilibrium point is asymptotically stable?

Group - B

(10 Marks)

Answer any ***two*** questions : 2×5=10

2. Explain Allee effect with physical interpretation.

3. Develop the epidemic SIR model.

4. Determine the stability of equilibrium point of the prey-predator system.

5. Using the travelling wave transformation convert Fisher's equation to a two dimensional dynamical system and derive it's equilibrium point.

Group - C

(18 Marks)

Answer any ***two*** questions : 2×9=18

6. Outline the Chemostat model for the growth of micro-organism. Determine the stability of the equilibrium point of the Chemostat model. 9

7. Consider the following system :

$$\frac{dx}{dt} = x(4 - x - y)$$

$$\frac{dy}{dt} = y(8 - 3x - y)$$

representing the change in densities of two competing species x and y. Find corresponding equilibrium points. Determine the stability of each equilibrium point and state their nature. 9

8. (a) The population dynamics of a species is governed by the discrete model

$$N_{t+1} = N_t \exp\left\{ r\left(1 - \frac{N_t}{K}\right)\right\}$$

Where r, K are positive constants.

(i) Determine the steady states and their stability nature.

(ii) Show that a period-doubling bifurcation occurs at r = 2.

(b) A drug is administered every six hours. Let D_n be the amount of drug in the blood system at n-th interval. The body eliminates a certain fraction p of the drug during each time interval. If the initial blood administered is D_0, find D_n and $\lim_{n\to\infty} D_n$. 4+5