U.G. 3rd Semester Examination 2021

MATHEMATICS (Honours)

Paper : DC-5

[Real Analysis]

(CBCS)

Full Marks : 32 Time : 2 Hours

The figures in the margin indicate full marks.
Notations and symbols have their usual meanings.

Group - A

(4 Marks)

1. Answer any ***four*** questions : 4×1=4

(a) Prove that, a banded monotonic function is a function of banded variation.

(b) Examine the convergence of the improper integral $\int_0^1 \frac{dx}{x^2}$.

(c) Applying Dirichlet's test determine the convergence of $\int_0^\infty \sin x^2 dx$

(d) If f is continous and positive on [a, b], then show that $\int_a^b f\,dx$ is also positive.

(e) State M_n-test for convergence of sequnce of functions.

(f) Let f be the function defined on [0, 2] by $f(x) = \begin{cases} 0, & x = \frac{n}{n+1} \\ 1, & \text{elsewhere} \end{cases}$

Is f integrable on [0, 2]?

(g) Show that the Cauchy Principle value of $\int_{-1}^{1} \frac{dx}{x^3}$ exists.

Group - B

(10 Marks)

Answer any *two* questions : 2×5=10

2. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ and g is bounde on $[a, b]$, then show that the series $\sum_{n=1}^{\infty} g(x) f_n(x)$ is uniformly convergent on $[a, b]$. 5

3. A function $f : [0, 1] \rightarrow \mathbb{R}$ is defined by $f(x) = (-1)^{n-1}$ when $\frac{1}{n+1} < x \le \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$ and $f(0) = 0$. Prove that f is integrable on $[0, 1]$ and $\int_0^1 f(x)dx = log\left(\frac{4}{e}\right)$. 3+2

4. If a function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$, then show that it is bounded on $[a, b]$ and $|f(x)| \le |f(a)| + V_f[a, b]$ for all $x \in [a, b]$. 2+3

5. Show that the integral $\int_0^{\frac{\pi}{2}} \log(\sin x) dx$ is convergent and hence evaluate it. 5

Group - C

(18 Marks)

Answer any *two* questions : 9×2=18

6. (a) If f is continuous on $[a, b]$ and if f' exists and is boundedin the interior, say $|f'(x)| \le K$ for all x in (a, b), then show that f is of bounded variation on $[a, b]$. Also show that boundedness of the derivative of f is not necessary for f to be of bounded variation. 5

(b) Discuss the convergency of the sequence of function $f_n(x)=\frac{nx}{1+n^2x^2}$. Where $x\in[0,1]$. 4

7. (a) Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$. Show that $|f|$ is also Riemann integrable on $[a,b]$. Is the converse true? Justify your answer. 4+2

(b) Examine whether Fundamental theorem of Integral Calculus is applicable to evaluate the integral $\int_0^3 f(x)\,dx$ where $f(x)=x[x], x\in[0,3]$. 3

8. (a) Obtain the Fourier series expansion of $f(x)$ in $[-\pi,\pi]$ where $f(x)=\begin{cases}0, & -\pi\le x<0\\ \frac{\pi x}{4}, & 0\le x\le\pi\end{cases}$. Hence show that the sum of the series $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$ is $\frac{\pi^2}{8}$. 4+2

(b) Let $X\subset\mathbb{R}$ and for each $n\in N$, $f_n:X\to\mathbb{R}$ be continuous on X. If $\{f_n\}$ converges uniformly to f on X then show that f is continuous on X. 3
