2023

GOUR MAHAVIDYALAYA

DEPARTMENT OF MATHEMATICS

Paper : MTMH - DC-03

[CBCS]

Full Marks: 32 Time: Two Hours

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings.

Group-A

4 Marks

1. Answer any **four** questions : [4×1=4]

(a) State Density property of $\mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the condition $|f(x) - f(y)| \leq (x - y)^2$ for all x, $y \in \mathbb{R}$. Prove that f is constant function on $\mathbb{R}$.

(c) Test the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

(d) Show by an example that every bounded sequence may not be a convergent sequence.

(e) Show that the set of all perfect square integers is a denumberable set.

(f) Give an example of a function f which is continuous on a point of it's domain but not differentiable.

(g) Reduce Rolle's theorem from Lagrange's mean value theorem.

Group-B

10 Marks

Answer any **two** questions : [2×5=10]

2. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$. [5]

3. Prove that an absolutely convergent series is convergent. Is the converse true? Justify your answer. [3+2]

4. Let $a_0, a_1, \ldots, a_n$ be real numbers satisfying the relation $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + + \frac{a_n}{n+1} = 0$. Show that the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ has at least one real root in $(0, 1)$. [5]

5. (a) Let f be the function defined on $\mathbb{R}$ by

$$\begin{aligned} f(x) &= x^2 + 1, \text{ if } x \in \mathbb{Q} \\ &= x, \text{ if } x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$

Prove that f has a discontinuity of the second kind at every point c in $\mathbb{R}$. [3]

(b) Let Ψ be a Lipschitz function defined on a domain D. Show that Ψ is uniformly continuous on D. [2]

Group-C

18 Marks

Answer any **two** questions : [2×9=18]

6. (a) Prove that $(1 + \frac{1}{x})^x > (1 + \frac{1}{y})^y$ if $x, y \in \mathbb{R}$ and $x > y > 0$. [4]

(b) Show that $f(x) = [x]$, where $[x]$ is the greatest integer function has a jump discontinuity at each integral value of x, the jump being 1. [2]

(c) Prove that the function g defined on $\mathbb{R}$ by $g(x) = \frac{1}{x^2+1}$; $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. [3]

7. (a) Test the convergence of the series $1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2^2.3} + \frac{1}{2^2.3^2} + \frac{1}{2^3.3^2} + \cdots$. [3]

(b) Let f be a function defined on $\mathbb{R}$ by

$$\begin{aligned} f(x) &= \frac{x}{1 + e^{\frac{1}{x}}}, \text{ if } x \neq 0 \\ &= 0, \text{ if } x = 0. \end{aligned}$$

Show that f is not differentiable at 0. [3]

(c)Prove that $\lim_{n\to\infty} \frac{\{(2n+1)(2n+2)-(2n+n)\}^{\frac{1}{n}}}{n} = \frac{27}{4e}$. [3]

8. (a) Show that 1 and -1 are the limits of $A = \{(-1)^m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$. [2]

(b) Let S be a subset of $\mathbb{R}$. Show that S is closed set if and only if $S = \overline{S}$. [3]

(c) Show that $\{1 + \frac{1}{n}\}^{n+1}$ is a monotone decreasing sequence. Also find it's limit. [4]

GOUR MAHAVIDYALAYA

MATHEMATICS (Honours)

Paper Code: MATH-DC04

Semester-II

Internal Examination

Time : 2 hour　　　　Full Marks : 32

Group-A

1. Answer any **four** questions.　　　　$4 \times 1 = 4$

(a) Find the order of each element of U_{12}.

(b) Give an example of a non-cyclic commutative group.

(c) Define $GL(n, \mathbb{R})$.

(d) Give an example of a finite ring with unity.

(e) Prove that in a Boolean ring R, $a + a = 0$ for every $a \in R$.

(f) Find all the zero divisor in the ring $\mathbb{Z}_{12}$.

(g) Show that the unity elements of a ring and its subring may be different by the help of an example.

Group-B

Answer any **two** questions.　　　　$2 \times 5 = 10$

2. Prove that every that every group of prime order is cyclic. Is the converse true? Justify your answer with an example. [5]

P.T.O.

3. Let $\phi : (G, \circ) \to (G, *)$ is a homomorphism. Prove that $Ker\phi$ is a normal subgroup of G. [5]

4. In the ring $\mathbb{Z}[i]$, Show that $I = \{a + bi \in \mathbb{Z}[i] | a, b \text{ are even}\}$ is an ideal of $\mathbb{Z}[i]$ but not a maximal ideal of $\mathbb{Z}[i]$. [5]

5. Define simple ring. Prpve that the set $M_2(\mathbb{R})$, the ring of all 2×2 over the field of real numbers is simple. [5]

Group-C

Answer any **two** questions.　　　　$2 \times 9 = 18$

6. (a) Show that the direct product $\mathbb{Z}_6 \times \mathbb{Z}_4$ of the cyclic group $\mathbb{Z}_6$ and $\mathbb{Z}_4$ is not a cyclic group. [4]

(b) i. Prove that a subgroup H of a group G is normal if and only if $aHa^{-1} = H$ for every a in G.

ii. Prove that a subgroup $K = \{\rho_0, \rho_1\}$ is not a normal subgroup of S_3. [3+2]

7. (a) Prove that the ring $(\mathbb{Z}_n, +, .)$ is an integral domain if and only if n is prime. [4]

(b) Prove that in ring R with unity an ideal M is a maximal ideal if and only if the quotient ring R/M is a field. [5]

8. (a) If G be a finite commutative group and d be a positive divisor of $o(G)$ then show that G has a subgroup of order d. [4]

(b) Prove that the set of all matrices $\left\{ \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ forms a field under matrix addition and multiplication. [5]

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