2020

MATHEMATICS (Honours)

Paper : MTMH - DC-3

[CBCS]

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings.

Group - A

(4 Marks)

1. Answer any *four* questions : $1\times4=4$

(a) State the Archimedean property of real numbers.

(b) Test whether the set $S = \{(x, y) \in \mathbb{R} : x^2 + y^2 < 1\}$ is closed or not.

(c) Find one limit point of the sequence $\{(-1)^n\}$.

(d) Is the series $\sum_{n=1}^{\infty} e^{\frac{1}{n}}$ convergent? Justify.

(e) Give an example of a function f which is nowhere continuous but $|f|$ is continuous everywhere.

(f) Let I be a non-trivial interval and $f : I \to \mathbb{R}$ be a differentiable function. State under what condition f is increasing on I.

(g) Reduce Rolle's theorem from Lagrange's mean value theorem.

Group - B

(10 Marks)

Answer any *two* questions. 5×2=10

2. Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, be a function. Prove that f is continuous at a point $c \in A$ if and only if for all sequences $\{a_n\}$ from A with $\lim\limits_{n\to\infty} a_n = c$, we have that $\lim\limits_{n\to\infty} f(a_n) = f(c)$. 5

3. (a) If $f(0) = f'(0) = 0$ and $f''(x)$ exists in $0 \leq x \leq h$, then prove that $f(h) = \frac{1}{2}h^2 f''(c)$, $0 < c < h$. 3

   (b) Show that $f(x) = [x]$, where $[x]$ is the greatest integer function, has a jump discontinuity at each integral value of x, the height being 1. 2

4. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then prove by using the mean value theorem that f is constant on $[a, b]$. 3

   (b) Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{n^2}{n!}$. 2

5. (a) State Taylor's theorem with Cauchy's form of remainder.

   (b) Prove that the equation $e^{-x} + 2 = x$ has at least one real solution. 2+3

Group - C

(18 Marks)

Answer any *two* questions. 9×2=18

6. (a) If $S = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$, then

   (i) show that 0 is a limit point of S.

(ii) show that $\frac{1}{k}$ is a limit point of S for all $k \in \mathbb{N}$,

(iii) find S' (the derived set of S). 1+2+2

(b) Test the convergence of the series

$$\frac{1}{2^2 \log 2} - \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} - \cdots .$$

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7. (a) If $\{x_n\}$ is a Cauchy sequence in $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ is uniformly continuous function, then show that the sequence $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence or otherwise show that the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, $0 < x < 1$ is not uniformly continuous. 3+2

(b) Prove that if $\{a_n\}$ converges to l, then the sequence $\{x_n\}$, where $x_n = \frac{a_1+a_2+\cdots+a_n}{n}$, also converges to l. 4

8. (a) State and prove Lagrange's mean value theorem. Describe the theorem in h-θ form. 2+4+1

(b) Apply Lagrange's mean value theorem in h-θ form for the function $\sin x$ in $[0, \frac{\pi}{2}]$ and find θ. 2