

2020

MATHEMATICS (Honours)

Paper : MTMH - DC- 06

[CBCS]

Full Marks : 32

Time : Two Hours

The figures in the margin indicate full marks.

*Candidates are required to give their answers
in their own words as far as practicable.*

Notations and symbols have their usual meanings.

Group - A

1. Answer any **four** questions: 1 × 4 = 4
- (a) Find the values of k for which the set $\{(k, 6), (2, k)\}$ forms a basis for \mathbb{R}^2 .
 - (b) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation given by $T(x, y) = (x, 0)$. Find $\ker T$.
 - (c) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation given by $T(x, y) = (y, x)$. Find the dimension of range of T .
 - (d) Give an example of a linear operator T on \mathbb{R}^2 such that T has no eigenvalues.
 - (e) Let T be a linear operator on $\mathbb{P}_2(\mathbb{R})$ defined by $T(f(x)) = f'(x)$. Find the characteristic polynomial of T .
 - (f) Prove that in an inner product space V , $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in V$.
 - (g) Let T and U be self-adjoint operators on a finite dimensional inner product space V . If TU is self-adjoint, then prove that $TU = UT$.

Group - B

Answer any two questions.

5×2=10

2. (a) Prove that a nonempty subset S of a vector space V over the field F is a subspace of V if and only if the following condition holds.
- (i) $x + y \in S, \forall x, y \in S$.
 - (ii) $\lambda x \in S, \forall x \in S$ and $\lambda \in F$. [3]
- (b) Let \mathbf{B} be a basis for the finite dimensional inner product space V . If $x \in V$ is such that $\langle x, z \rangle = 0, \forall z \in \mathbf{B}$, then prove that $x = \theta$. [2]
3. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear operator given by $T(x, y, z) = (0, 0, x)$. Find the matrix representation of T with respect to the standard ordered basis for \mathbb{R}^3 . Also determine whether T is diagonalizable or not. [2+3]
4. Let $S = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ be a subset of the inner product space \mathbb{R}^3 . Apply Gram-Schmidt orthogonalization process on S to obtain an orthonormal basis for $\text{span}(S)$. [5]
5. Let $M_2(\mathbb{R})$ be the set of all 2×2 real matrices and $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + b = 0 \right\}$ and $T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : c + d = 0 \right\}$. Find the dimension of S, T and $S \cap T$. [5]

Group - C

Answer any two questions.

9×2=18

6. (a) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by $T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t)$. Find a basis for
- (i) the range of T
 - (ii) the kernel of T .
- Also find their dimensions. [2+2+1]

- (b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by $T(x, y, z) = (2x + 3y + 5z, 4x - 5y + 6z, 5x + 7y + 2z)$. Find the matrix representation of T relative to the basis $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. [4]
7. (a) For the subsets $A = \{x, y, z\}$ and $B = \{x + y, y, y + z, x + z\}$ of the vector space \mathbb{R}^3 , prove that $\text{span}(A) = \text{span}(B)$. [3]
- (b) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator given by $T(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$, where $0 < \theta < \pi$. Determine whether T is
- (i) self-adjoint
 - (ii) unitary
 - (iii) normal. [2+2+2]
8. (a) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (3x + y, x + 3y)$. Determine whether the vectors $v_1 = (1, -1)$ and $v_2 = (1, 1)$ are eigenvectors of T . [2]
- (b) In an inner product space V , prove that $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in V$. [5]
- (c) Prove that every eigenvalue of a self-adjoint operator T on a finite dimensional inner product space V is real. [2]
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