

UG/2nd Sem (H)/23/(CBCS)

2023

MATHEMATICS (Honours)**Paper Code : MTMH DC-3****(Real Analysis-I)**

Full Marks : 32

Time : Two Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A**(4 Marks)**Answer any *four* questions : $1 \times 4 = 4$

1. (a) Is the set $\{1/n : n \in \mathbb{N}\}$ closed in \mathbb{R} ? Include or exclude the minimum number of points in it to obtain a closed set in \mathbb{R} .

- (b) Let $\{x_n\}$ be a sequence in \mathbb{R} such that

$$x_1 = 1 \text{ and } x_{n+1} = \left[1 - \frac{1}{(n+1)^2} \right] \cdot x_n, \quad n \geq 1.$$

What do you think $\lim_{n \rightarrow \infty} x_n$ is?

P.T.O.

(c) Give an example of a convergent series $\sum u_n$ for which the series $\sum u_n^2$ diverges.

(d) Let $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right)$$

for all $x \in \mathbb{R} \setminus \{0\}$. Extend the function f to \mathbb{R} by suitably defining at 0 such that it is continuous at 0.

(e) If $a \in (0, \infty)$, check whether or not that \exists a natural number n such that $\frac{1}{n} < a < n$.

(f) Use the identity $1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$ for $x \neq 1$ to arrive at an expression for the sum $1 + x + 2x^2 + \dots + nx^n$.

(g) Is the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{x}{x^2 + 1}$$

for all $x \in \mathbb{R}$ monotonically increasing?

Group - B

(10 Marks)

Answer any two questions. $5 \times 2 = 10$

2. Prove that a set in \mathbb{R} is open if and only if it can be expressed as the union of countably many disjoint open intervals in \mathbb{R} . 5

3. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$ for all $n \geq 1$ converges to the positive root of $x^2 - x - 7 = 0$. 5

4. (a) Let S and T be two non-void bounded subsets of \mathbb{R} . Prove that,

$$\sup(A \cup B) = \max\{\sup A, \sup B\}. \quad 3$$

(b) Expand $f(x) = \log(1+x)$; $-1 < x \leq 1$ in an infinite series. 2

5. Let $f: [a, b] \rightarrow \mathbb{R}$ be differentiable at $\alpha \in (a, b)$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - y| < \delta$ and $a < x \leq \alpha \leq y < b$, then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(\alpha) \right| < \epsilon. \quad 5$$

P.T.O.

(4)

Group - C

(18 Marks)

Answer any two questions.

9×2=18

6. (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) := x|x| \text{ for all } x \in \mathbb{R}.$$

- (i) Show that f is differentiable everywhere on \mathbb{R} .
- (ii) Determine the derived function f' .
- (iii) Is the derived function f' continuous?
- (iv) Is the derived function f' differentiable?

2+1+1+1

(b) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in \mathbb{R} such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. Then show that the sequence

$$\left\{ \frac{x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1}{n} \right\}$$

is convergent. Find its limit.

3+1

7. (a) Prove each of the following :

- (i) If $I \subset \mathbb{R}$ is an interval and $f: I \rightarrow \mathbb{R}$ is continuous and nowhere vanishing on I , then either $f > 0$ on I or $f < 0$ on I .

(5)

(ii) If f is a continuous function taking values in \mathbb{Z} or in \mathbb{Q} , then f is a constant function.

(iii) If $f: [a, b] \rightarrow \mathbb{R}$ is a non-constant continuous function, then $f([a, b])$ is uncountable.

2+2+2

(b) Prove that every absolutely convergent infinite series in \mathbb{R} is convergent. Give an example of an infinite series in \mathbb{R} which is convergent but not absolutely convergent.

2+1

8. (a) Let A be a set in \mathbb{R} defined by

$$A := \left\{ \frac{(-1)^n n}{n+1}; n \in \mathbb{N} \right\}$$

(i) Find the limit points of A .

(ii) Is A a closed set?

(iii) Is A an open set?

(iv) Does A contain any isolated point?

(v) Find the closure of A .

1+1+1+1+1

(b) Construct a continuous function from $(0, 1)$ onto $[0, 1]$. Can such a function be one-to-one? 3+1

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2023

MATHEMATICS (Honours)**Paper Code : MTMH DC-4****(Abstract Algebra)**

Full Marks : 32

Time : Two Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

The notations and symbols have their usual meaning.

Group - A**(4 Marks)**

1. Answer any *four* questions : 1×4=4

(a) Prove or disprove : "Every non-trivial group has a non-trivial cyclic subgroup". 1

(b) Let $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ be two

elements of S_3 . Find the solution of the equation $ax = b$ in S_3 . 1

(c) Find all generators of \mathbb{Z}_{20} . 1

P.T.O.

(2)

(d) Find the order of the element $(\bar{1}, \bar{3})$ in $\mathbb{Z}_3 \times \mathbb{Z}_6$.

1

(e) Let $(G, *)$ be a group and $a, b \in G$. Suppose that $a^2 = e$ and $a * b * a^{-1} = b^7$. Prove that $b^{48} = e$, where e is the identity element in G .

1

(f) Let a and b be two elements in a field F . If $a^2 = b^2$, then prove that either $a = b$ or $a = -b$.

1

(g) Find the order of each element of the group U_{12} .

1

Group - B

(10 Marks)

Answer any two questions. $5 \times 2 = 10$

2. State and prove Cayley's theorem for groups. 1+4

3. State first isomorphism theorem for groups. Using this, prove that any two finite cyclic groups of same order are isomorphic and any infinite group is isomorphic to \mathbb{Z} . 1+4

4. Prove that a field is an integral domain. Give an example of an integral domain which is not a field. 4+1

5. Let R be a commutative ring with unity. Then show that an ideal M is maximal ideal if and only if R/M is a field. 5

Group - C

(18 Marks)

Answer any two questions. $9 \times 2 = 18$

6. (a) Let $a, b \in \mathbb{R}$ and consider a mapping $f_{a,b}: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_{a,b}(x) = ax + b$ for all $x \in \mathbb{R}$. Let

$$G = \{f_{a,b} : a, b \in \mathbb{R}, a \neq 0\}.$$

Prove that G forms a group with respect to usual composition of two mappings. 3

- (b) Let $(H, *)$ be a subgroup of a group $(G, *)$. Define $aHa^{-1} = \{aha^{-1} : h \in H\}$. Show that $(aHa^{-1}, *)$ is a subgroup of G . Also prove that $|H| = |aHa^{-1}|$. 2+1

- (c) For a ring R , define $M_R = \left\{ \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} : a, b \in R \right\}$.

Show that M_Q forms a field but M_R does not. 3

P.T.O.

7. (a) State and prove Lagrange's theorem. 1+2
- (b) If H is a subgroup of a commutative group G , then show that G/H is commutative. Is the converse true? Justify your answer. 2+1
- (c) Let G be a finite commutative group of order n and $\gcd(m, n) = 1$. Prove that $\varphi: G \rightarrow G'$ defined by $\varphi(x) = x^m$ is an isomorphism. 3
8. (a) Prove that, every non-zero element of \mathbb{Z}_n is a unit or a divisor of zero. 3
- (b) Show that the characteristic of an integral domain is either zero or a prime number. Give an example of an integral domain, which has an infinite number of elements, yet it is of finite characteristic. 2+1
- (c) Let F be a field of order 2^n . Prove that $\text{char } F = 2$. 3
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