2022

MATHEMATICS (Honours)

Paper Code : MTMH DC-5

(Real Analysis-II)

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

(4 Marks)

1. Answer any *four* questions : 1×4=4

(a) Is the function given by

$$f(x)=\begin{cases} x^2\cos\frac{1}{2} & \text{if } 0<x\le 1 \\ 0 & \text{if } x=0 \end{cases}$$

a function of bounded variation? Justify your answer.

(b) Define Refinement of a partition.

(c) Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

P.T.O.

(d) If $f:[0,1]\rightarrow\mathbb{R}$ defined by

$$f(x) = 0,\ x\in[0,1]\cap Q$$
$$= 1,\ x\notin[0,1]\cap Q$$

then show that f is not R-integrable on [0, 1].

(e) Show that the improper integral $\int_0^1 \frac{dx}{1-x}$ is divergent.

(f) Let $f_n(x)=xe^{-nx}, x\geq 0$. Show that the sequence of function $\{f_n\}$ is point wise convergent on $[0,\infty)$ to the function f defined by $f(x)=0, x\geq 0$.

(g) Explain why the Fundamental theorem of integral calculus can not be used to evaluate $\int_0^3 x[x]dx$.

Group - B

Answer any *two* questions : 5×2=10

2. Prove that $\lim_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$

3. Let $f_n(x)=\log(n^2+x^2),\ x\in\mathbb{R}$. Show that the sequence $\{f_n'\}$ is uniformly convergent on $\mathbb{R}$ but the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$.

4. Prove that $f:[a,b]\rightarrow\mathbb{R}$ be a function of bounded variation on [a, b] iff f can be expressed as the difference of two monotonic increasing functions on [a, b].

5

5. Let $f:[a,b]\to\mathbb{R}$ be integrable on $[a, b]$. If there exists a positive real number K such that $f(x)\geq K$ for all $x\in[a,b]$ then show that $\frac{1}{f}$ is integrable on $[a, b]$.

Group - C

Answer any two questions : 9×2=18

6. (a) Let $f:[a,b]\to\mathbb{R}$ be boundend on $[a, b]$ and let f be continuous on $[a, b]$ except on a infinite subset $S\subset[a, b]$ such that the number of limit points of S is finite. Then prove that f is *R-integrable* on $[a, b]$. 4

   (b) Prove that the even function $f(x)=|x|$ on $[-\pi,\pi]$ has as cosine series in Fourier's form as

   $$\frac{\pi}{2}-\frac{4}{\pi}\left\{\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right\}.$$ 5

7. (a) If $f:[a,b]\to\mathbb{R}$ be integrable on $[a, b]$ and f possesses an antiderivative ϕ on $[a, b]$, then prove that $\int_a^b f=\phi(b)-\phi(a)$

   [fundamental theorem of Integral Calculus]. 5

   (b) Find the length of the perimeter of the cardioid $r=a(1+\cos\theta)$. 4

8. (a) Test the convergence of β function. 5

   (b) State and prove the Cauchy-Hadamard theorem. 4

2022

MATHEMATICS (Honours)

Paper Code : MTMH DC-6

(Linear Algebra)

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

(4 Marks)

1. Answer any *four* questions : 1×4=4

(a) Find $k \in \mathbb{R}$ so that the set $S = \{(1, 2, 1), (k, 3, 1), (2, k, 0)\}$ is linearly dependent in $\mathbb{R}^3$.

(b) Find the dimension of the subspace S of $\mathbb{R}^4$ defined by

$$S = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$$

(c) If α, β be two orthogonal vectors in a Euclidean space V, then show that

$$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$$

P.T.O.

(d) Find the range of the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x,y,z)=(x+z, x+y+2z, 2x+y+3z)$$

(e) If $\{u_1, u_2, ..., u_r\}$ be an orthonormal set, prove that for any $v \in V$, the vector $w = v - <v, u_1> u_1 - <v, u_2> u_2 - ... - <v, u_r> u_r$ is orthogonal to each of the u_i.

(f) Let λ be an eigenvalue of a linear operator T on an inner product space V. If $T^* = T^{-1}$, then show that $|\lambda| = 1$.

(g) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$$

Then find rank T.

Group - B

(10 Marks)

Answer any *two* questions. 5×2=10

2. If U and W be two subspaces of a vector space V over a field F such that $U \cap W = \{\theta\}$ and if $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $\{\beta_1, \beta_2, ..., \beta_n\}$ be respectively the bases of U and W, then show that $\{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ is a basis of $U + W$. 5

3. Determine the linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors (0, 1, 1), (1, 0, 1), (1, 1, 0) of $\mathbb{R}^3$ to (1, 1, 1), (1, 1, 1), (1, 1, 1) respectively. Verify that dimker T + $dim\ ImT = 3$. 5

4. Find the algebraic and geometric multiplicities of each eigen value of the matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$ 5

5. If $T:V \to V$ be a linear transformation, show that the following statements are equivalent :

(i) Range $T \cap KerT = \{0\}$

(ii) If $T(T(v)) = 0$ then $T(v) = 0, v \in V$. 3+2

Group - C

(18 Marks)

Answer any *two* questions. 9×2=18

6. (a) If V is a finite dimensional inner product space and W is a subspace of V, then show that $V = W \oplus W^{\perp}$. 5

(b) Let T be a normal operator. Prove :

(i) $T(v) = 0$ if and only if $T^*(v) = 0$

(ii) $T - \lambda I$ is normal. 2+2

P.T.O.

7. (a) Extend the set of vectors {(2, 3, −1), (1, −2, −4)} to an orthogonal basis of the Euclidean space $\mathbb{R}^3$ with standard inner product and then find the associated orthonormal basis. 5

(b) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1+x_3, -2x_1+x_2, -x_1+2x_2+4x_3)$. Show that T is invertible. 4

8. (a) Apply gram-schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, spanned by the vectors (1, 1, 0, 1), (1, −2, 0, 0), (1, 0, −1, 2). 5

(b) A linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x_1, x_2, x_3) = (x_2+x_3,\ x_3+x_1,\ x_1+x_2,\ x_1+x_2+x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find ker T. Verify that the set $\{T(\in_1), T(\in_2), T(\in_3)\}$ is linearly independent in $\mathbb{R}^4$, where $\in_1=(1,0,0)$, $\in_2=(0,1,0)$ and $\in_3=(0,0,1)$. 4

UG/3rd Sem (H)/22/(CBCS)

2022

MATHEMATICS (Honours)

Paper Code : MTMH DC-7

(Multivariate Calculus & Vector Calculus)

Full Marks : 32

Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Symbols used in this question paper bears their original meaning unless stated.

Group - A

(4 Marks)

1. Answer any *four* questions : 1×4=4

(a) Show that $\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} f(x, y)$ does not exist, where

$$f(x,y)=\begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

(b) If $f(x,y)=\begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2=0, \end{cases}$

then show that $f_x(0,0) \neq f_y(0,0)$.

P.T.O.

(c) Find the directional derivative of

$\varphi = 4xz^3 - 3x^2y^2z$ at (2, −1, 2) in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

(d) Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} dr d\theta$.

(e) Evaluate $\iiint_V dxdydz$ where V is the tetrahedron bounded by $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

(f) Prove that $Curl(\text{grad}\ \varphi) = \bar{0}$

(g) State Gauss divergence theorem.

Group - B

(10 Marks)

Answer any *two* questions : 5×2=10

2. Show that the necessary and sufficient condition that a nonzero differentiable vector function $\bar{f}(t)$ to possess the constant magnitude is that $\bar{f}.\frac{d\bar{f}}{dt} = 0$.

3. Evaluate the line integral $\oint_\Gamma \bar{f}.d\bar{r}$ by Stokes theorem where Γ being the boundary of the rectangle $0 \le x \le \frac{\pi}{2}$, $0 \le y \le 2$, $z = 1$ and $\bar{f} = \sin z\hat{i} - \cos x\hat{j} + \sin y\hat{k}$.

4. Show that $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

has directional derivative at (0, 0) in any direction $\beta = (l, m)$ where $l^2 + m^2 = 1$ but f is discontinuous at (0, 0).

5. Show that if $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the maximum value of xyz is $abc/3\sqrt{3}$.

Group - C

(18 Marks)

Answer any *two* questions : $9 \times 2 = 18$

6. (a) State and prove Schwarts Theorem. 6

(b) Show that $\int_0^1 \int_0^{1-y^2} \left\{ (x-1)^2 + y^2 \right\} dx\, dy = \frac{44}{105}$. 3

7. (a) Show that $\vec{\nabla} \log r = \frac{1}{r^2} \vec{r}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$. 4

(b) Use Greens theorem to evaluate $\oint_\Gamma \left(xy\, dx + y^2 dy \right)$ where Γ is a square in the xy plane with vertices (0, 0), (0, 1), (1, 0) and (1, 1). 5

P.T.O.

8. (a) Let $V = \sin^{-1}\sqrt{\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}}$. Prove that

$$x^2 \frac{\partial^2 V}{\partial x^2} + 2xy \frac{\partial^2 V}{\partial x \partial y} + y^2 \frac{\partial^2 V}{\partial y^2} = \frac{\tan V}{12}\left(\frac{13}{12} + \frac{\tan^2 V}{12}\right).$$

5

(b) If $\vec{f} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{g} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$ then prove that $\dfrac{d}{dt}\left(\vec{f} \times \dfrac{d\vec{g}}{dt}\right) = \hat{i} + 6\hat{j} + 2\hat{k}$ at $t = 1$.

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