CONNECTEDNESS AND COMPACTNESS IN TOPOLOGY

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Contents

[1] Application of both Connectedness and Compactness is added

1 Introduction to Topology:

Topology is a branch of mathematics that studies the properties of spaces that are preserved under continuous deformations, such as stretching, twisting, and bending, but not tearing or gluing. It focuses on the qualitative aspects of geometry rather than the exact shape and size of objects. The fundamental concepts in topology include:

1. Topological Spaces: These are sets equipped with a topology, which is a collection of open sets that satisfies certain axioms, allowing the definition of concepts like continuity, convergence, and compactness.

2. Homeomorphisms: These are continuous functions between topological spaces that have continuous inverses, indicating that two spaces are topologically equivalent.

3. Continuous Functions: Functions that preserve the structure of a topological space, meaning the preimage of any open set is open.

4. Compactness and Connectedness: Key properties of topological spaces. A space is compact if every open cover has a finite subcover, and connected if it cannot be divided into two disjoint non-empty open sets.

Topology has applications across various fields, including algebra, analysis, and geometry, and is foundational in understanding the global properties of spaces used in many areas of mathematics and science.

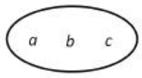
In this thesis we will focused on **Compactness and Connectedness**. Before that, let's start with some basic concepts that are useful in our study.

Definition: A topology is a set of collection of a set X satisfying following conditions:
(i) $\emptyset, X$ belongs to $\mathcal{T}$
(ii) Arbitrary union of elements of $\mathcal{T}$ belongs to $\mathcal{T}$
(iii) Finite intersection of elements of $\mathcal{T}$ belongs to $\mathcal{T}$

The set X on which the topology $\mathcal{T}$ is defined together with the topology is called as topological space.
i.e., here $(X, \mathcal{T})$ is a topological space.

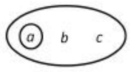
Let us take some examples of collection of subsets of $X = \{a, b, c\}$ and see whether the are a topology or not.

Examples:
(i)



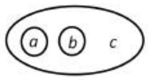
Let $\mathcal{T}_1 = \{\emptyset, X\}$, Then $\mathcal{T}_1$ is a topology on X.

(ii)



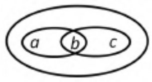
Let $\mathcal{T}_2 = \{\emptyset, X, \{a\}\}$, then $\mathcal{T}_2$ is a topology on X.

(iii)



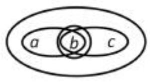
Let $\mathcal{T}_3 = \{\emptyset, X, \{a\}, \{b\}\}$, then $\mathcal{T}_3$ is not a topology on X, as $\{a\}, \{b\} \in \mathcal{T}_3$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_3$.

(iv)



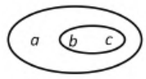
Let $\mathcal{T}_4 = \{\emptyset, X, \{a, b\}, \{b, c\}\}$, then $\mathcal{T}_4$ is also not an topology on X, as $\{a, b\}, \{b, c\} \in \mathcal{T}_4$ but $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_4$.

(v)



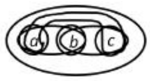
Let $\mathcal{T}_5 = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$, then $\mathcal{T}_5$ is a topology on X as adding the set $\{b\}$ in the example-(iv) solve the problem for being a topology.

(vi)



Let $\mathcal{T}_6 = \{\emptyset, X, \{b, c\}\}$, then $\mathcal{T}_6$ is a topology on X.

(vii)



Let $\mathcal{T}_7 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$, then $\mathcal{T}_7$ is a topology on X.

1.1 Some definitions of useful terminologies:

Discrete topology: Let X be any set. The collection of all subsets of X is a topology on X. This topology is called as discrete topology.
From the previous examples, $\mathcal{T}_7$ is the discrete topology on the set $X = \{a, b, c\}$.

Indiscrete topology: Let X be any set. The collection which consists only $\emptyset$ and X is a topology on X. This topology is called as indiscrete topology or trivial topology.
From the previous examples, $\mathcal{T}_1$ is the discrete topology on the set $X = \{a, b, c\}$.

Open set in topology: Let $(X, \mathcal{T})$ is a topological space. All the members of $\mathcal{T}$ is known as open sets or $\mathcal{T}$-open.
i.e, A subset U of X is said to be an open set or $\mathcal{T}$-open if $U \in \mathcal{T}$.

Closed set in topology: Let $(X, \mathcal{T})$ is a topological space. Then complements of all open sets are known as closed sets or $\mathcal{T}$-closed set.
i.e, A subset F of X is said to be closed or $\mathcal{T}$-closed if $X - F \in \mathcal{T}$.

Clopen set: Let $X, \mathcal{T}$ be a topological space. A subset of X is said to be clopen if the subset is both open and closed together.

For any set X, $\emptyset, X$ are clopen for any topology on X.

Finer and Coarser: Let $\mathcal{T}, \mathcal{T}'$ are two topologies on a set X. If $\mathcal{T} \subseteq \mathcal{T}'$, we

say $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is coarser than $\mathcal{T}'$.
If we have $\mathcal{T} \subset \mathcal{T}'$, we can use strictly finer and strictly coarser for $\mathcal{T}'$ and $\mathcal{T}$ respectively.

1.2 Basis of a topology:

Definition: Let X be a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called as basis element) such that
(i) For each $x \in X$, there is atleast one basis element B which contains x.
i.e, $\forall x \in X, \exists\, B \in \mathcal{B}\ :\ x \in B$.
(ii) If $x \in B_1 \cap B_2$; $B_1, B_2 \in \mathcal{B}$ then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

If $\mathcal{B}$ satisfies the above two conditions that is $\mathcal{B}$ is a basis of a topology, then we can define a topology $\mathcal{T}$ generated by $\mathcal{B}$ as:
$U \in \mathcal{T}$ if for each $x \in U \subseteq X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$

Again we are going to take some collection of subsets of the set $X = \{a, b, c\}$ and check whether they are a basis of topology on X.
Examples:
(i) $\mathcal{B}_1 = \{\emptyset, X\}$
(ii) Let $\mathcal{B}_2 = \{\{a\}, \{b\}, \{b, c\}\}$, then $\mathcal{B}_2$ is a basis of a topology on X as it satisfies the above two conditions.
(iii) Let $\mathcal{B}_3 = \{\{a\}, \{b\}, \{a, b, c\}\}$, then $\mathcal{B}_3$ is a basis of a topology on X as it satisfies the conditions of basis.
(iv) Let $\mathcal{B}_4 = \{\{a\}, \{b\}\}$, then $\mathcal{B}_4$ is not a basis of a topology on X as $c \in X$ but $\nexists B \in \mathcal{B}_4\ : c \in B$ ($\mathcal{B}_4$ does not satisfy condition (i)).
(v) Let $\mathcal{B}_5 = \{\{a, b\}, \{a, c\}\}$, then $\mathcal{B}_5$ is not a basis of a topology on X as $\{a, b\} \cap \{a, c\} = \{a\}$ but there is no basis element which is a subset of $\{a\}$ ($\mathcal{B}_5$ does not satisfy condition (ii)).
(vi) Let $\mathcal{B}_6 = \{\{a\}, \{a, b\}, \{a, c\}\}$, then $\mathcal{B}_3$ is a basis of a topology on X as it satisfies the conditions of basis.
(vii) Let $\mathcal{B}_7 = \{\{a\}, \{b\}, \{c\}\}$, then $\mathcal{B}_3$ is a basis of a topology on X as it satisfies the conditions of basis.

Note: Let X be any set. The collection of all singleton (one point) subsets of X is a basis of the disctete topology on X
e.g, From the above examples of bases, $\mathcal{B}_7$ is the basis of the discrete topology on $x = \{a, b, c\}$.

1.3 Some popular topological spaces:

Standard topology: Let $X = \mathbb{R}$. If $\mathcal{B}$ be the collection of all open intervals in the real line, i.e, $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$. Then the topology generated by $\mathcal{B}$ is called as Standard topology or usual on $X = \mathbb{R}$.
It is denoted as $\mathbb{R}_u$.
It is important to note that we can define usual topology on $\mathbb{R}^n \ \forall n \in \mathbb{N}$ by using all the open balls as a basis.

Lower limit topology: Let $X = \mathbb{R}$. If $\mathcal{B}'$ be the collection of all semi-open intervals of the form $[a, b) = \{x \in \mathbb{R} \ : a \leq x < b\}$. Then the topology generated by $\mathcal{B}'$ is called lower limit topology on $\mathbb{R}$.
It is denoted as $\mathbb{R}_l$
Here $\mathcal{B}' = \{[a, b) \ : a, b \in \mathbb{R}$.

Upper limit topology: Let $X = \mathbb{R}$. If $\mathcal{B}''$ be the collection of all semi-open intervals of the form $(a, b] = \{x \in \mathbb{R} \ : a < x \leq b\}$. Then the topology generated by $\mathcal{B}''$ is called upper limit topology on $\mathbb{R}$.
Here $\mathcal{B}'' = \{(a, b] \ : a, b \in \mathbb{R}$.

K-topology: Let $X = \mathbb{R}$. Let $K = \{\frac{1}{n} \ : n \in \mathbb{Z}^+\}$ and $\mathcal{B}^*$ be the collection of all open intervals (a, b) with all the sets of the form $(a, b) - K$. Then the topology generated by $\mathcal{B}^*$ is called K- topology on $\mathbb{R}$.
It is denoted as $\mathbb{R}_K$.
Here $\mathcal{B}^* = \{(a, b), (a, b) - K \ : a, b \in \mathbb{R}\}$.

Product topology: Let X and Y be two topological spaces. The product topology on $X \times Y$ is a topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is open subset of X and V is open subset of Y.

Order topology: Let X be a set with a simple order relation. Assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:
(i) all open intervals $(a.b)$ in X,
(ii) all the intervals of the form $[a_0, b)$, where a_0 is the smallest element of X (if any).
(iii) all the intervals of the form $(a, b_0]$, where b_0 is the largest element of X (if any).

Co-finite topology: Let X be any set. A collection of all subsets U of X such that $X - U$ is finite or all of X, this collection is a topology on X, called as Co-finite topology or finite complement topology.
Co-finite topology generally written as $\mathcal{T}_f$.
i.e, $\mathcal{T}_f = \{U \subseteq X \ : U = \emptyset \text{ or } X - U \text{ is finite}\}$

Subspace topology: Let $(X, \mathcal{T})$ be a topological space. If Y is a subset of X, The collection $\mathcal{T}_Y = \{U \cap Y \ : U \in \mathcal{T}\}$ forms a topology on Y, called as subspace topology or relative topology.
$(Y, \mathcal{T}_Y)$ is called subspace of $(X, \mathcal{T})$.

E.g, Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{b\}\}$.
Let $Y = \{a, c\} \subset X$
Now,

$$\begin{aligned} \mathcal{T}_Y &= \{U \cap Y \ : U \in \mathcal{T}\} \\ &= \{\emptyset, Y, \{a\}, \{c\}\} \end{aligned}$$

Metric topology: If (X, d) be a metric space. Then the collection of all $\epsilon-$ball $B_d(x, \epsilon)$ forms a basis for a topology on X. This topology is called topology on the metric space X, induced by the metric d.

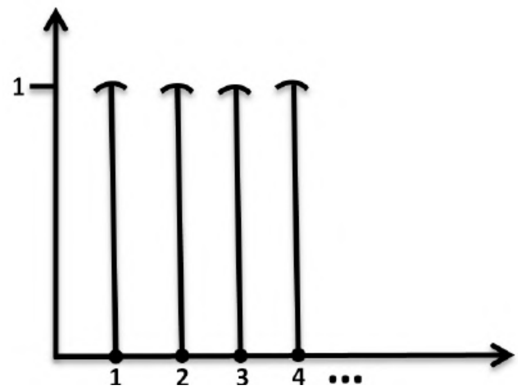
1.4 Linear continuum:

Least upper bound property: An ordered set A is said to have the least upper bound property if every subet A_0 of A that is bounded above, has a least upper bound.

Example:
(i) Consider $A = (-1, 1)$, then A has least upper bound property.
(ii) Consider $B = (-1, 0) \cup (0, 1)$, a subset of $\mathbb{R}$.
Let $B_0 = (-1, 0)$, then $B_0 \neq \emptyset$ and B_0 has upper bound (say 0.5) in B and 0 is the least upper bound of B_0 but $0 \notin B$.
So, B does not have least upper bound property.

Linear continuum: A simply ordered set L having more than one element is called a linear continuum if the following conditions holds:
(i) L has the least upper bound property,
(ii) If $x < y$, $\exists z \in L : x < z < y$, where $x, y \in L$.

Examples: $\mathbb{R}, \mathbb{Q}$ are linear continuum.
Another linear continuum is $\mathbb{Z}^+ \times [0, 1)$, shown in the below figure.



We have used the dictionary order in this topological space.

2 Continuity of a function:

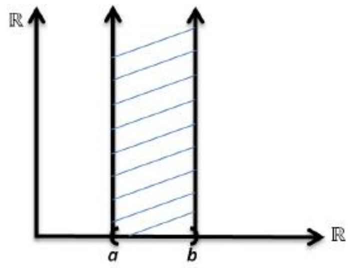
Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be continuous if for each open set V of Y, the set $f^{-1}(V)$ is open in X.
$f^{-1}(V) = \{x \in X : f(x) \in V\}$.

Continuity of a function f depends not only on the function f but also on the topologies for its domain and range. So we can use continuity relative to topologies on X and Y.

Propositions:
(i) A function $f : X \to Y$ is continuous if and only if the inverse of each member of a basis $\mathcal{B}$ for Y is open subset of X.
(ii) A function $f : X \to Y$ is continuous if and only if the inverse of each member of a sub-basis $\mathcal{S}$ for Y is an open subset of X.

Example: The projection mappings from $\mathbb{R}^2$ to $\mathbb{R}$ are both continuous.
As, Let $\Pi_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Pi_1 ((x, y)) = x$$



Now,

$$\begin{aligned}\Pi_1^{-1}((a,b)) &= (a,b)\times\mathbb{R}\\ &= (a,b)\times(-\infty,\infty)\end{aligned}$$

Which is open in $\mathbb{R}\times\mathbb{R}$
Hence Π_1 is continuous.
Similarly, Π_2, which is defined by $\Pi_2((x,y)) = y$ is continuous.

Theorem: Let X and Y be two topological spaces with $f : X \to Y$. Then the following statesments are equivalent.
(i) f is continuous.
(ii) For every open subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$
(iii) For every closed set B of Y, $f^{-1}(B)$ is closed in Y.
(iv) For all $X \in X$ and for all neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Proof: At first, we will show that $(i) \implies (ii)$.
Given, $f : X \to Y$ be a continuous map. Let A be a subset of X and $x \in A$.
Now, $x \in A \implies f(x) \in f(\overline{A})$.
We have to show that $f(x) \in \overline{f(A)}$.
Let V be a neighbourhood of $f(x)$, i.e, V is an open set in Y containing $f(x)$. Therefore, $f^{-1}(V)$ is open in X and contains x.
Then $f^{-1}(V)$ intersects A at some point y and hence $f(x) \in \overline{f(A)}$.
$\therefore f(\overline{A}) \subseteq \overline{f(A)}$.

Now, we will show that $(ii) \implies (iii)$
Let B is closed set in Y and $A = f^{-1}(B)$.
Now $f(A) = f(f^{-1}(B)) \subseteq B$.
Let, $x \in \overline{A}$, then

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B \qquad \text{,As } B \text{ is closed.}$$

i.e,

$$f(x) \in B \implies x \in f^{-1}(B) = A \implies x \in A$$

Therefore, $\overline{A} \subseteq A$ And hence $A = f^{-1}(B)$ is closed in X.

Now, we will show that, $(iii) \implies (i)$.
Let V be an open set in Y and $B = Y - V$.
Then

$$\begin{aligned} f^{-1}(B) &= f^{-1}(Y) - f^{-1}(V) \\ &= X - f^{-1}(V) \end{aligned}$$

Since, B is closed in Y then $f^{-1}(B)$ is closed in X.
i.e, $X - f^{-1}(V)$ is closed in X.
i.e, $f^{-1}(V)$ is open in X.
Therefore, for all open V in Y, $f^{-1}(V)$ is open in X. Hence, f is continuous.

Now, we will prove $(i) \implies (iv)$
Given, f is a continuous map.
Let $x \in X$ and V is a neighbourhood of $f(x)$.
Then $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) = f(f^{-1}(V)) \subseteq V$.
Since, x and V are arbitrary, then $\forall \;\; x \in X$ and for all neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Lastly, we will show that $(iv) \implies (i)$.
Let V be an open set in Y and $x \in f^{-1}(V)$, then $f(x) \in V$.
Then there exists a neighbourhood U_x of x such that $f(U_x) \subseteq V$ and thus

$$U_x \subseteq f^{-1}(V) \implies f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$$

So, $f^{-1}(V)$ is an open set as arbitrary union of open sets is open.
Therefore, for all open V in Y, $f^{-1}(V)$ is open in X.
Hence, f is continuous.

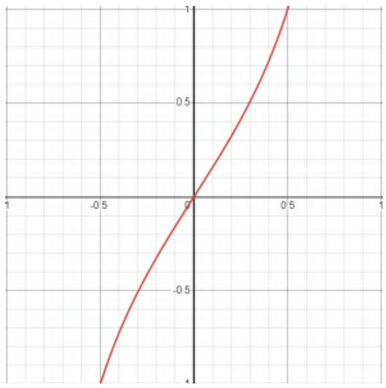
3 Homeomorphism:

Definition: Let X and Y be two topological spaces such that $f : X \to Y$ be a bijection, if both the function $f : X \to Y$ and inverse function $f^{-1} : Y \to X$ are continuous, then f is called homeomorphism.

In simple words, A homeomorphism is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open in Y if and only if U is open in X.

A function f is called bi-continuous or topological if f is open and continuous. Therefore, f is homeomorphism if and only if f is continuous and bijective.

Example: We can show that an open interval of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.
Let $X = (-1, 1)$ and $f : X \to \mathbb{R}$ defined by $f(x) = tan\frac{\pi x}{2}$



The above graph represents the function.
Here, f is an one-one, onto and a bijective function. Also f^{-1} is continuous.
Hence, f is a homeomorphism and therefore X is homeomorphic to $\mathbb{R}$
i.e, $(-1, 1) \simeq \mathbb{R}$.

Graphically, two spaces are homeomorphic if we can get one space from the anothe without join it or cut it, we can use stretching or bending. To understand this let's see some examples.

Examples:
(i) A circle with radius 1 is homeomorphic to any other circle of any radius.
(ii) Coffee cup $\simeq$ Doughnut
(iii) There is no homeomorphism between a line and a circle.

Topological property: A property P is called topological property or topological invariant if whenever a topological space $(X, \mathcal{T})$ has the property P, every topological space homeomorphic to $(X, \mathcal{T})$ also has that property P.

Let us check some property of a topological spaces and further check whenever they are topological property or not.

Examples:
(i) We know that $X = (-1, 1)$ and $\mathbb{R}$ are homeomorphic and length of $X \neq$ length of $\mathbb{R}$
Therefore, length is not a topological property.

(ii) Let $X = \mathbb{R}^+$ and $f : X \to X$ defined by $f(x) = \frac{1}{x}$. Then $f^{-1}(x) = \frac{1}{x}$ and both $f,\ f^{-1}$ is continuous.
So, f is a homeomorphism.
Now, $(a_n) = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ is a cauchy sequence but $(f(a_n)) = \{1, 2, 3, 4, \cdots\}$ is not a cauchy sequence.
Hence, Cauchy sequence is not a topological property.

As a conclusion it can be added here, that
(i) Length, boundedness, cauchy sequence, radius of a circle, completeness are not topological property.
(ii) Connectedness, compactness, first countable, second countable, metrizable are topological property.

4 Connectedness:

Connectedness is a fundamental concept in topology that describes the idea of a space being "in one piece." A topological space is said to be connected if it cannot be divided into two disjoint non-empty open subsets. In other words, there is no way to split the space into two separate parts without cutting it.

Key points about connectedness include:

1. Connected Spaces: A space is connected if the only subsets that are both open and closed (clopen) are the empty set and the entire space itself.

2. Path Connectedness: A stronger form of connectedness where any two points in the space can be joined by a continuous path. Every path-connected space is connected, but not all connected spaces are path-connected.

3. Components: The maximal connected subsets of a space. Any space can be decomposed into its connected components, which are disjoint and cover the entire space.

4. Applications: Connectedness is crucial in many areas of mathematics, such as in the study of continuous functions, where the image of a connected space under a continuous function is also connected.

Understanding connectedness helps in analyzing the structure and behavior of topological spaces, aiding in the comprehension of more complex topological properties and theorems.

Let's describe all these terms in more describing way and take a deep dive into this topic.

Separation: Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X.
i.e, If $(X, \mathcal{T})$ be a topological space, a separation is a pair U, V satisfying the four conditions,
(i) $U \neq \emptyset, \;\; V \neq \emptyset$
(ii) $U, V \in \mathcal{T}$
(iii) $U \cap V = \emptyset$
(iv) $U \cup V = X$.

Connected space: A topological space X is said to be connected if there does not exist any sepataion of X.

Connected is a topological property.
i.e, If X is a connected space, then all the spaces homeomorphic to X is connected.

Theorem: A space X is connected if and only if the only subsets of X are both open and closed in X are $\emptyset$ and X.
Proof: Let X is a connected space. It is obvious that $\emptyset$ and X are both open and closed in X.
If possible let A be a non-empty proper subset of X which is both open and closed in X.
Then $X - A$ is also a non-empty proper subset of X and $A, \; X - A$ are disjoint.
As A is both open and closed, its complement i.e, $X - A$ is also both open and closed.
Therefore, $A, \; X - A$ be two non-empty disjoint open subsets of X such that $A \cup (X - A) = X$.
So, $A, \; X - A$ forms a separation of X, which contradicts the fact that X is connected.
Hence, there can not exists any such non-empty proper clopen subset of X.
Hence, the only subsets of X that are both open and closed are X and $\emptyset$.

Conversely, let $\emptyset, \; X$ are the only both open and closed subsets of X.
If possible, let X is not connected. Then there exists two non-empty disjoint open subsets A, B of X, such that $A \cup B = X$
$\therefore B = X - A$.
Since A is open, B is closed and hence B is both open and closed, which is a contradiction as we found a non-empty proper subset B which is both open and closed.

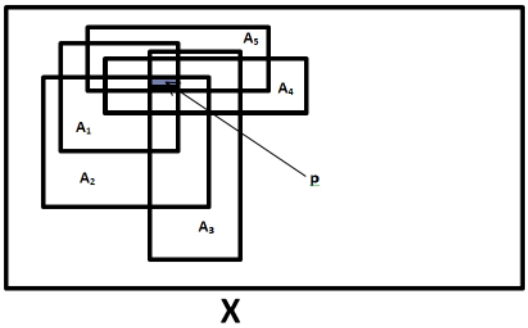
So our assumption is wrong, i.e, X is connected.

Lemma: If Y is a subspace of X, a separation of Y is a pair of non-empty disjoint sets A and B whose union is Y, neither of which contains a limit point of other. The space Y is connected if there exists no separation of Y.
Proof: Let A and B forms a separation of Y, then A is both open and closed in Y.
The closure of A in Y is $\bar{A} \cap Y$, where $\bar{A}$ is the closure of A in X.
Since, A is closed in Y, then $A = \bar{A} \cap Y$.
So, $\bar{A} \cap B = \emptyset$
$\therefore B$ does not contain any limit point of A.
Similarly, it can be proven that A does not contain any limit point of B.

Conversely, let A and B be two disjoint non-empty sets whose union is Y and neither of which contains a limit point of other.
Therefore $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$
Then $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$.
$\therefore A,\ B$ are both closed in Y and also $A = X - B,\ B = X - A$, implies that A, B are both open in Y with $A \neq \emptyset, B \neq \emptyset,\ \ A \cap B = \emptyset$ and $A \cup B = Y$.
Hence, A, B forms a separation of Y.

Theorem: If the sets C and D forms a separation of X and if Y is a connected subspace of X, then Y entirely lies either in C or in D.
Proof: Given, C and D forms a separation of X. Then $C, D \neq \emptyset,\ C \cap D = \emptyset,\ C \cup D = X$ and C, D are open in X.
If possible let Y lies in both C and D.
So, $C \cap Y,\ D \cap Y$ are open in Y.
Then $C \cap Y,\ D \cap Y$ are two non-empty disjoint open subsets of Y whose union is Y.
Therefore, $C \cap Y,\ D \cap Y$ forms a separation of Y, Which contradicts the fact that Y is connected.
Therefore, either Y lies entirely in C or Y lies entirely in D.

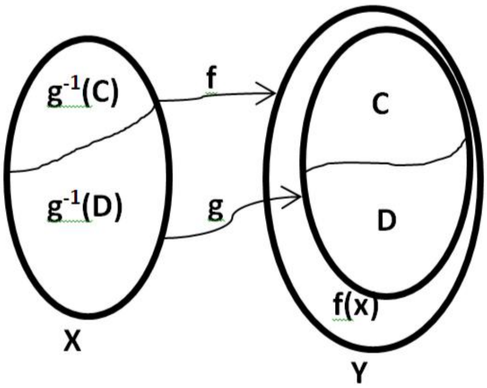
Theorem: The union of a collection of connected subspaces of a topological space X that have a point in common, is connected.
Proof:



Let $\{A_\alpha \ : \alpha \in \Lambda\}$ be a collection of connected subspaces of X, having a common point p.
i.e, $p \in \cap_{\alpha\in\Lambda} A_\alpha$
Let $Y = \cup_{\alpha\in\Lambda} A_\alpha$
If possible let Y is not connected, i.e, there is a separation of Y. Let C, D forms a separation of Y.
As C, D forms a separation, then $C \cap D = \emptyset$. Which implies, either $p \in C$ or $p \in D$.
Without loss of generality, let $p \in C$.
As $p \in A_\alpha \ \ \forall \alpha \in \Lambda$ and for all α, A_α is a connected subspace of X,
So $A_\alpha \subseteq C \ \ \forall \alpha \in \Lambda$
$\implies \cup_{\alpha\in\Lambda} \subseteq C$
$\implies Y \subseteq C$.
$\therefore D = \emptyset$, which contradicts that C, D forms a separation of Y.
$\therefore \cup_{\alpha\in\Lambda} A_\alpha$ is connected.

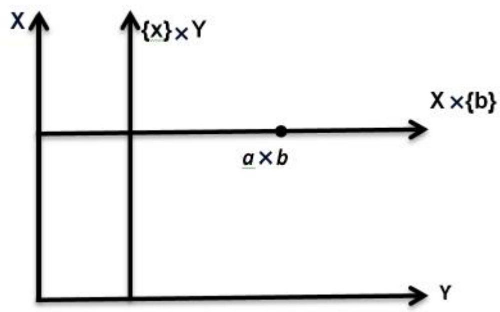
Theorem: Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.
Proof: Given, A is a connected subspace of X and $A \subset B \subset \bar{A}$.
If possible, let B is not connected. Then let C, D forms a separation of B.
As $A \subset B$ and A is connected, then A lies either in C or in D.
Without loss of generality, we are assuming that A lies in C, i.e, $A \subset C$.
Then $\bar{A} \subset \bar{C}$
Since $\bar{C} \cap D = \emptyset$, then $B \cap D = \emptyset$, as $B \subset \bar{A} \subset \bar{C}$.
As $D \subset B$ and $B \cap D = \emptyset$, so $D = \emptyset$, which contradicts that C, D forms a separation of B.
Therefore, B is connected.

Theorem: The image of a connected space under a continuous map is connected.
Proof:



Let X is a connected space and $f : X \to Y$ be a continuous function.
Let $Z = f(X)$ and then consider $g : X \to Z$, which is a continuous map as f is continuous.
If possible let Z is not connected. Then let C, D forms a separation of Z.
Since C, D are open in Z, $g^{-1}(C), g^{-1}(D)$ are open in X, as g is continuous.
As C, D are disjoint, then $g^{-1}(C), g^{-1}(D)$ are also disjoint.
As g is surjective, $g^{-1}(C), g^{-1}(D)$ are non-empty.
Also, $g^{-1}(C) \cup g{-}1(D) = X$
Therefore, we find that $g^{-1}(C), g^{-1}(D)$ forms a separation of X. Which is a contradiction as X is connected.
So, our assumption is wrong and hence $Z = f(X)$ is connected.

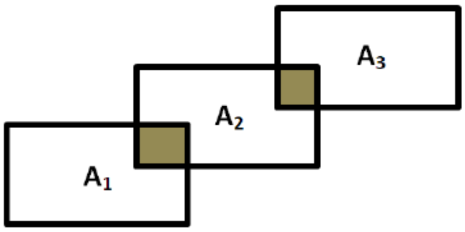
Theorem: A finite cartesian product of connected spaces is connected.
Proof:



Let us consider two connected spaces X and Y. Also let a point $a \times b$ in the space $X \times Y$.
Then the space $X \times \{b\}$ is connected as $X \times \{b\}$ is homeomorphic to a connected

space X.
Similarly $\{x\} \times Y$ is connected as it is homeomorphic to a connected space Y, for any $x \in X$.
Then the space $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected, as it is union of two connected spaces have the point $x \times b$ in common.
Now, $\cup_{x \in X} T_x$ is connected as it is union of connected spaces having $a \times b$ as a common point.
Also, $\cup_{x \in X} T_x$ is all of $X \times Y$.
Therefore, $X \times Y$ is a connected space.
We can prove this for finite product of connected spaces by induction by using the fact that $(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times X_2 \times \cdots \times X_n)$.

Theorem: Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Show that, $\cup A_n$ is connected.
Proof:



Given, (A_n) be a sequence of connected subspaces of X and $A_n \cap A_{n+1} \neq \emptyset \ \forall n$.
$\therefore A_1 \cap A_1 \neq \emptyset$
Therefore, A_1, A_2 are two connected subspaces of$X$ having a common point, which implies that $A_1 \cup A_2$ is connected.
Again let us consider $A_1 \cup A_2$ and A_3, both are connected subspaces of X and both have a common point as $A_2 \cup A_3 \neq \emptyset$. So, $(A_1 \cup A_2) \cup A_3$, i.e, $A_1 \cup A_2 \cup A_3$ is connected.
Proceeding in this manner, we can show that $\cup A_n$ is connected.

Another way of proving: If possible let A, B be a separation of the space $\cup A_n$.
As A_1 is a connected subspace of $\cup A_n$, then either $A_1 \subseteq A$ or $A_1 \subseteq B$.
Without loss of generality, let us assume that $A_1 \subseteq A$.
Now, as $A_1 \cap A_2 \neq \emptyset$, then $A_2 \subseteq A$.
Again, as $A_2 \cap A_3 \neq \emptyset$, then $A_3 \subseteq A$.
Similarly, $A_3 \subseteq A, \ A_4 \subseteq A \cdots \ A_n \subseteq A$ and so on.
$\therefore \cup A_n \subseteq A$ and hence $B = \emptyset$, which shows there can not be exist any separation.
Hence, $\cup A_n$ is connected.

4.1 Totally disconnected:

A space is totally disconnected if its only connected subspaces are one-point (trivial) sets.

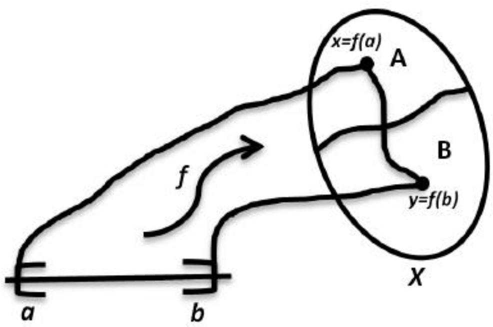
Ex: Prove that, if X has a discrete topology, then X is totally disconnected.
Proof: Given $(X, \mathcal{T}_d)$ is a topological space, where $\mathcal{T}_d$ is the discrete topology on X.
Let $U \neq \emptyset$ be any subset of X.
Case-1: Let U has more than one element and $x \in U$.
Then consider two sets $\{x\}$ and $U - \{x\}$.
Now, $\{x\} \neq \emptyset$, $U - \{x\} \neq \emptyset$.
Also, $\{x\} \cap (U - \{x\}) = \emptyset$ and $\{x\} \cup (U - \{x\}) = U$.
Again, in discrete topology, {x} and $U - \{x\}$ are open.
So, $\{x\}$, $U - \{x\}$ forms a separation of U and thus U is not connected.
Hence subsets of discrete topological spaces having more than one elements are disconnected.
Case-2: Let U has exactly one point, say x, i.e, $U = \{x\}$
Therefore, we can not find two non-empty disjoint subset of $U = \{x\}$. Hence, $U = \{x\}$ is connected.
Thus, only connected subsets of discrete topological spaces are the one point sets.
$\therefore (X, \mathcal{T}_d)$ is totally disconnected.

Note: However, the converse part is not true.
i.e, A totally disconnected space can not always be discrete topological space.
E.g, Let X be an infinite set with finite completement topology (co-finite topology), say $\mathcal{T}_f$.
$\therefore \mathcal{T}_f = \{U \subset X : U = \emptyset \text{ or } X - U \text{ is finite }\}$.
For any finite subset U of X, U is not open in X. So, $\mathcal{T}_f$ is not discrete topology.
Now let us consider a two point subset Y of X such that $Y = \{x, y\}$.
Then $X - \{y\} = A$ (say) and $X\{x\} = B$ (say) are two open sets in $\mathcal{T}_f$.
Therfore $A \cap Y$, $B \cap Y$ are two non-empty open subsets of Y.
Also, $(A \cap Y) \cap (B \cap Y) = \emptyset$ and $(A \cap Y) \cup (B \cap Y) = Y$.
Thus, $A \cap Y$ and $B \cap Y$ forms a separation of Y.
Therefore Y is not connected.
In this manner we can prove all the sets having more than one point in the co-finite topological space is not connected.
Hence, only connected subsets of the co-finite topological space X are the single point sets, i.e, X is totally disconnected.

4.2 Path connected:

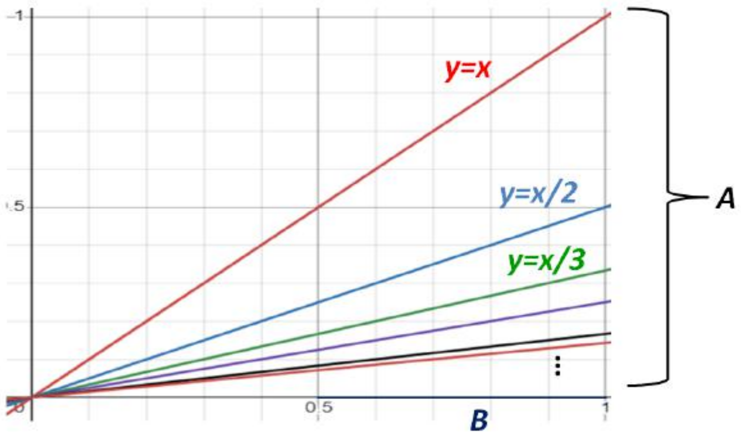
Given points x and y of the space X. A path in X from X to y is a continuous map $f : [a, b] \to X$ of some closed intervals in $\mathbb{R}$ into X such that $f(a) = x$ and $f(b) = y$. A set X is said to be path connected if every pair of points of X can be joined by a path.

Theorem: Any path connected space is connected.
Proof:



Let X be a path connected space, i.e, every pair of points in X is can be joined by a path.
If possible, let X is not connected. Then there exists a separation of X, say A, B be a separation of X.
Therefore, A, B be two non-empty,disjoint, open subsets of X such that $A \cup B = X$.
As, X is path connected, so let $f : [a, b] \to X$ be any path in X, where $f(a) \in A,\ f(b) \in B$.
Since, image of a connected set is connected under a continuous map, $f([a, b])$ is connected.
$\therefore f([a, b])$ either lies in A or in B.
Therefore, there is no such path to join a point in A to a point in B, which is a contradiction as X is path connected.
Hence, X is connected.

Note: Every connected set may not be path connected.
e.g, Consider the following subsets of $\mathbb{R}^2$,
$A = \{(x, y) : 0 \leq x \leq 1; y = \frac{x}{n}; n \in \mathbb{N}\}$
$B = \{(x, 0\} : \frac{1}{2} \leq X \leq 1\}$



As A is a union of arbitrary connected lines having a common point $(0,0)$, A is connected.
Also B is connected as it is an interval on $\mathbb{R}$.
Here, each point of B is a limit point of A.
Therefore, $A \cup B$ has no separation, i.e, $A \cup B$ is connected.
But there does not exist any path to join a point in A and a point in B.
Hence, $A \cup B$ is not path connected.

For this example, if we take $B = \{(x,0) : 0 \leq x \leq 1\}$, then $A \cup B$ will be a path connected space.
So for this example of proving $A \cup B$ is connected but not path connected, the maximum range of B should be $B = \{(x,0) : 0 < x \leq 1\}$.

Intermediate value theorem: Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$
Proof: Assume the hypotheses of the theorem. The sets

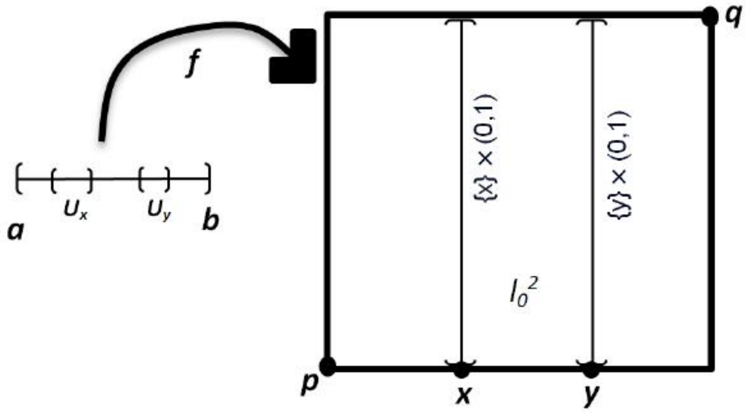
$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cap (r, +\infty)$$

are disjoint, and they are nonempty because one contains $f(a)$ and the other contains $f(b)$. Each is open in $f(X)$, being the intersection of an open ray in Y with $f(X)$. If there were no point c of X such that $f(c) = r$, then $f(X)$ would be the union of the sets A and B. Then A and B would constitute a separation of $f(X)$, contradicting the fact that the image of a connected space under a continuous map is connected.

Note: The intermediate value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Let us take some more examples of connected space which is not path connected.

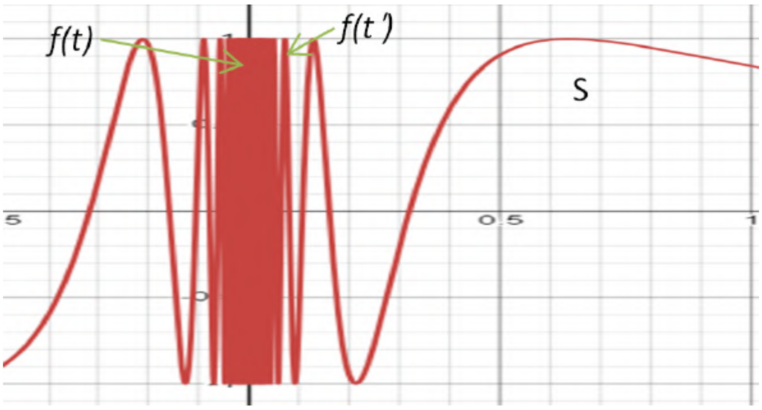
Example-1:



We previously know about l_0^2. We are going to prove this topological space as not path connected.
l_0^2 satisfies the condition of linear continuum and hence it is a linear continuum, which implies l_0^2 is connected.
Let $p = 0 \times 0$ and $q = 1 \times 1$.
Let us assume that l_0^2 is path connected. Then there exists a path $f : [a, b] \to l_0^2$ with $f(a) = p,\ f(b) = q$.
Since f is continuous then by Intermediate value theorem, $f([a, b])$ must contain every point $x \times y$ of I_0^2.
Then for each $x \in I,\ f^{-1}(x \times (0, 1)) = U_x$ (say).
Then U_x is non-empty open subset of $[a, b]$ as f is continuous. Then for a fixed $x \in I$, let a rational number $q_x \in U_x$ in in such a way that if x, y are two disjoint element in I, then U_x and U_y are also disjoint.
Therefore, if we take an map $x \to q_x$, it is an injective mapping of I into $\mathbb{Q}$.
This contradicts the fact that the interval I is uncountable.
So, our assumption is wrong and hence l_0^2 is not path connected.

Example-2: Let us take another example of connected space which is not path connected.

Let $S = \{x \times sin(\frac{1}{x}) : 0 < x \leq 1\}$



Since S is the image of connected set $(0, 1]$ under a continuous map, then S is connected.
Then also $\bar{S}$ (closure of S in $\mathbb{R}^2$) is also connected.
Here, $\bar{S} = S \cup (\{0\} \times [-1, 1])$.
If possible, let $\bar{S}$ is path connected. Then there exists a path $f : [a, c] \to \bar{S}$ beginning at the origin and ending at a point of S.
The set $\{t \in [a, c] : f(t) \in \{0\} \times [-1, 1]\}$ is closed as it is pre-image of the a closed set under a continuous map.
So, this set has the largest element, say b.
Then also $f : [b, c] \to \bar{S}$ is also a path which maps b into $\{0\} \times [-1, 1]$ and all the other elements in S.
Now, for any $\delta > 0$,
$b < \frac{2}{(4n-3)\pi} < b + \delta$ and $b < \frac{2}{(4n-1)\pi} < b + \delta$ for some $n \in \mathbb{N}$.
Let $f(t) = (\gamma_1(t), \gamma_2(t))$, where

$$\gamma_1(t) = t \; \forall t \in (b, c]$$
$$\gamma_2(t) = sin(\frac{1}{t}) \; \forall t \in (b, c]$$
$$\gamma_1(b) = 0, \quad \gamma_2(b) \in [-1, 1]$$

Let $t_1 = \frac{1}{(4n-3)\pi} > b$, then $\gamma_2(t_1) = sin(\frac{(4n-3)\pi}{2}) = 1$
Again, let $t_2 = \frac{1}{(4n-2)\pi} > b$, then $\gamma_2(t_2) = sin(\frac{(4n-1)\pi}{2}) = -1$, where $b < t_1, t_2 < b + \delta$
Now, $|\gamma_2(t_1) - \gamma_2(t_2)| = |1 - (-1)| = 2$
Which implies that γ_2 is not continuous, which contradicts the fact that f is contin-

uous.
So, our assumption is wrong.
Hence, $\bar{S}$ is not connected.

4.3 Components:

If $(X, \mathcal{T})$ is a topological space, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x, y.
The equivalence class are called components or connected components of X.

Let us take a topological space as example to understand the components.
Example: Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, X, \{2\}, \{1, 3\}\}$. Let us check the components of X.
Here, $\{2\}$, $\{1, 3\}$ forms a separation of X, hence X is not connected.
Let us check all the subspace of X for their connectedness.
$Y_1 = \{1\}$, then $\mathcal{T}_{Y_1} = \{U \cap Y_1 : U \text{ is open in } X\} = \{\emptyset, Y_1\}$, is connected.
$Y_2 = \{2\}$, then $\mathcal{T}_{Y_2} = \{U \cap Y_2 : U \text{ is open in } X\} = \{\emptyset, Y_3\}$, is connected.
$Y_3 = \{3\}$, then $\mathcal{T}_{Y_3} = \{U \cap Y_3 : U \text{ is open in } X\} = \{\emptyset, Y_3\}$, is connected.
$Y_4 = \{1, 2\}$, then $\mathcal{T}_{Y_4} = \{U \cap Y_4 : U \text{ is open in } X\} = \{\emptyset, Y_4, \{1\}, \{2\}\}$, is not connected.
$Y_5 = \{2, 3\}$, then $\mathcal{T}_{Y_5} = \{U \cap Y_5 : U \text{ is open in } X\} = \{\emptyset, Y_5, \{2\}, \{3\}\}$, is not connected.
$Y_6 = \{1, 3\}$, then $\mathcal{T}_{Y_6} = \{U \cap Y_6 : U \text{ is open in } X\} = \{\emptyset, Y_6\}$, is connected.
So, there are four connected subspaces of X as $Y_1 = \{1\}$, $Y_2 - \{2\}$, $Y_3 = \{3\}$, $Y_6 = \{1, 3\}$.
Since $Y_1, Y_3 \subseteq Y_6$, there are two components $\{2\}$ and $\{1, 3\}$ of the space X such that $X = \{2\} \cup \{1, 3\}$

Theorem: The components of a topological space X are connected disjoint subspace of X whose union is X, such that for each non-empty connected subspace of X intersects only one of them.
Proof: As components are equivalent class, then components of X are disjoint and their union is X.
Let A be a non-empty connected subspace of X. If possible let A intersects two components C_1 and C_2 of X atleast in one points, say x_1 and x_2 respectively.
i.e,

$$x_1 \in C_1 \cap A$$
$$x_2 \in C_2 \cap A$$

Then, $x_1 \sim x_2$, as x_1, x_2 belongs to a connected space A.
So, x_1, x_2 have same equivalence class but in different components. It can not be possible. The only possibility is $C_1 = C_2$
Now, the only proof left is to prove that a component C is connected.
Let $x_0 \in C$, then $\forall x \in C,\ \ x_0 \sim x$.
Then there exists a connected subspace A_x containing x_0, x and $A_x \subseteq C$.
Then $C = \cup_{x \in C} A_x$.
Since all A_x are connected with common point x_0, then the union is also connected.
Therefore, C is connected.
This completes the proof.

Path component: We define an equivalence relation on the space X by define, $x \sim y$ if there is a path in X from x to y. The equivalence classes are called path components.

Theorem: The path components of X are path connected disjoint subspace of X whose union is X such that every non-empty path connected subspace of X intersects only one of them.
Proof: As path components are equivalence class, then path components are disjoint and their union is X.
Let A be a non-empty path connected subspace of X.
If possible let A intersects two path components C_1 and C_2 of X atleast in one points, say x_1 and x_2 respectively.
i.e,

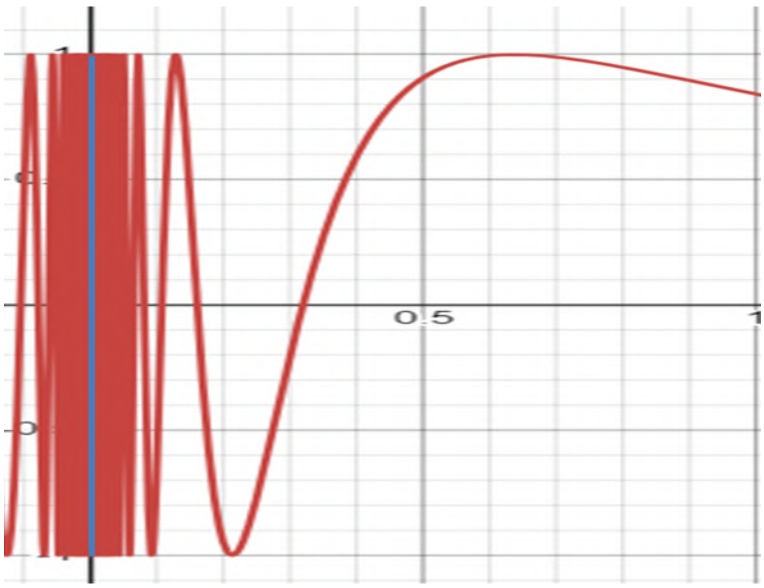
$$x_1 \in C_1 \cap A$$
$$x_2 \in C_2 \cap A$$

Then, $x_1 \sim x_2$, as x_1, x_2 belongs to a path connected space A.
So, x_1, x_2 have same equivalence class but in different path components. It can not be possible. The only possibility is $C_1 = C_2$
Now, the only proof left is to prove that a path component C is path connected.
Let $x_0 \in C$, then $\forall x \in C,\ \ x_0 \sim x$.
Then there exists a path connected subspace A_x containing x_0, x and $A_x \subseteq C$.
Then $C = \cup_{x \in C} A_x$.
Since all A_x are path connected with common point x_0, then the union is also path connected.
Therefore, C is path connected.
This completes the proof.

Note: Now there is a question from the above theorem can be raised.
We said here that, Every path coconnected subspace of X can only intersect one path

component of X.
Also, in the previous theorem we learned, every connected subspace of X can only intersect one component.
It is very clear that a path connected set is connected and a path component is a component of X.
So, the second theorem itself implies the first one. Now, what if we try to use the first condition in the second theorem?
i,e, Can a non-empty connected space intersect two path components?

For this check, we will try with an familiar example where the path connection was not established, which is the topologists sine curve.



As we have seen before, this is not path connected but connected. Moreway, there is two path components $\{0\} \times [-1, 1]$ and the rest of the set, i.e, the blue and red shades in the above figure.
But this set is connected. so if we take the whole set as A, a connected subspace, it intersects both its components.
That is why we used a path connected supspace instead of a connected subspace.

This topologist sine curve is very important because it can be used in various theories to verify or disown.

There is one more topic in connectedness and that is the locally connectedness. As like it names it conclude local connectedness.

Locally connected space: A topological space X is said to be locally connected at x if for every neighbourhood U of x, there exists a connected neighbourhood V of x contained in U.
If the space X is locally connected at each of it's points in X, then this space X is called the locally connected space.

4.4 Locally path connected space:

A topological space X is said to be locally path connected at x if for every neighbourhood U of x, there exists a path connected neighbourhood V of x contained in U.
If the space X is locally path connected at each of it's points in X, then this space X is called the locally path connected space.

Let us see through some examples whether they are connected,locally connected, both or neither.

Examples:
(i) Every discrete topological space X is locally connected but not connected if X has more than one points.
(ii) Every intervals and rays in the real line ($\mathbb{R}$) with respect to the usual topology are both connected and locally connected.
(iii) The subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is locally connected but not connected.
(iv) The topologists sine curve is not locally connected (As, we can not find a connected neighbourhood for the point $(0, v)$, where v is the largest point of the set $\{0\} \times [-1, 1]$) but connected.
(v) $\mathbb{Q}$ is neither connected nor locally connected.

Theorem: A space X is locally connected if and only if for every open set U of X, each component of U is open in X.
Proof: Suppose that X be locally connected. Let U be an open set in X and C be a component of U. Let $x \in C$, then $x \in U$.
Now, since X is locally connected, there exists a connected neighbourhood V of x such that $x \in V \subseteq U$.
Since $x \in V$ and $x \in C$ and also C is an component then $x \in V \subseteq C$. Which implies that C is open.
Therefore, C is open in X.

Conversely, suppose that components of all the open sets of X are open.
Let $x \in X$ be any point and U be a neighbourhood of x.
Let C be a component of U, which contains x.
Now from the assumption, C is connected and open in X.
As x is arbitrary, then $\forall x \in X$ and for all neighbourhood U of x, there exists a connected neighbourhood (open set) C if x such that $x \in C \subseteq U$.
Therefore, X is locally connected.

We have similar theorem for the locally path connected space.

Theorem: A space is locally path connected if and only if for every open set U of X, each path component of U is open in X.

the proof is exactly similar to the before, so we are skipping this.

Theorem: Prove that every component C of a space is closed.
Proof: Given, C is a component of a topological space. Then C is connected.
We know that C is connected $\implies \bar{C}$ is connected. Also $C \subseteq \bar{S}$
As C is a component i.e, C is some maximum connected set, then $C = \bar{C}$.
Therfore C is closed.

The relation between path components and components is given in the following theorem:

Theorem: If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.
Proof: Let C be a component of X; let x be a point of C; let P be the path component of X containing x. Since P is connected, $P \subset C$. We wish to show that if X is locally path connected, $P = C$. Suppose that $P \subsetneq C$. Let Q denote the union of all the path components of X that are different from P and intersect C : eacb of them necestarily lies in C, so that

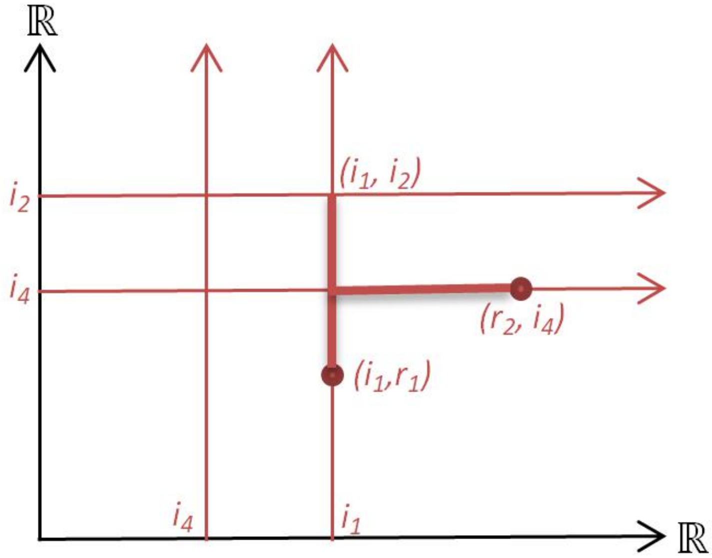
$$C = P \cup Q.$$

Because X is locally path connected, each path component of X is open in X. Therefore, P (which is a path component) and Q (which is a union of path components) are open in X, so they constitute a separation of C. This contradicts the fact that C is connected.
Hence, the theorem is proved.

4.5 More theories on connectedness:

1. With the help of connectedness property, we can prove that $(0, 1), (0, 1]$ and $[0, 1]$ all are not homeomorphic.
We are taking the sets $(0, 1)$ and $(0, 1]$. We are assuming that these two sets are homeomorphic.
Now if we remove the point 1 from the set $(0, 1]$, then it becomes $(0, 1)$ and it is still a connected set. But if we remove one any one point from the set $(0, 1)$, it turns into a disconnected set.
As connectedness is a topological space, i.e, all the homeomorphic sets to a connected set is connected, it contradicts that $(0, 1)$ and $(0, 1]$ are homeomorphic.
In this way, we can prove that $[0, 1]$ is also not homeomorphic to the above two sets by removing two points 0,1 from it.

2. We can prove that $\mathbb{R}^2 - A$ is a connected space for any countable set A.
For this we are taking A as the largest countable set in $\mathbb{R}^2$, i.e, $A = \mathbb{Q} \times \mathbb{Q}$.
Then $\mathbb{R}^2 - A = (\mathbb{Q}^c \times \mathbb{Q}^c) \cup (\mathbb{Q} \times \mathbb{Q}^c) \cup (\mathbb{Q}^c \times \mathbb{Q})$.
So, there can be three types of points in $\mathbb{R}^2 - A$ as $(i_1, i_2), (i_1, r_1), (r_2, i_4)$, where $i_1, i_2, i_3, i_4 \in \mathbb{Q}^c$ and $r_1, r_2 \in \mathbb{Q}$.

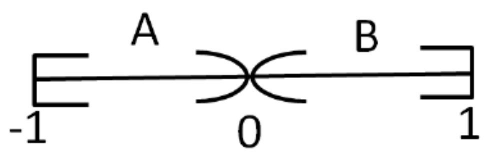


In the above image all the red lines are the path in the set $\mathbb{R}^2 - A$ and in the picture we can see that all the above points can be always connect through a path. If we keep changing the points we can see that union these lines (paths) will cover all of

$\mathbb{R}^2 - A$.
So, $\mathbb{R}^2 - A$ is path connected and hence connected.

Now, if we talk about general vesion of this theory, that is A is any countable set in $\mathbb{R}^2$.
Then for any point in $\mathbb{R}^2 - A$, there is uncountable number of lines passing through which does not intersects A.
For any two points in $\mathbb{R}^2 - A$, there is a pair of lines that do intersect each other but do not intersect A. So we can say, both the points are connected through a path.
Therefore $\mathbb{R}^2 - A$ is path connected and hence connected.

3. If we take a subspace $Y = [-1, 0) \cup (0, 1]$ of the topological space $X = \mathbb{R}$, then clearly Y has a separation. We are going to prove it theoritically.
Let $A = [-1, 0)$ and $B = (0, 1]$ be two subsets of Y.



Now, $A \neq \emptyset, B \neq \emptyset \quad A \cap B = \emptyset$ and $A \cup B = Y$
Here, we can skip the open set condition and instead of that we are going to use the condition of containing limit points.
i.e,

$$\begin{aligned} \bar{A} \cap B &= [-1, 0] \cap (0, 1] \\ &= \emptyset \\ A \cap \bar{B} &= [-1, 0) \cap [0, 1] \\ &= \emptyset \end{aligned}$$

Therefore, A, B be two non-empty,disjoint subset of Y, whose union is Y and neither of A, B contains a limit point of other.
$\therefore A, B$ forms a separation of Y and hence Y is not connected.

4. In the previous way we tried we can directly say that $Y = [-1, 1]$ is a connected subspace of $X = \mathbb{R}$.

As, to find a separation, we can try in a way like, $A = [-1, 0)$ and $B = [0, 1]$. Then A, B satisfies all the three conditions but

$$\begin{aligned}\bar{A} \cap B &= [-1, 0] \cap [0, 1] \\ &= \{0\} \\ &= \emptyset\end{aligned}$$

So, A, B can not form a separation of Y and hence Y is connected.

5. From the study of real analysis we know that the set of all rational number $\mathbb{Q}$ is densed but not connected due to the presence of irrational numbers.
Now let us prove this with the connectedness property of topological space.
Let $Y = \mathbb{Q}$ be a subspace of the topological space $X = \mathbb{R}$. Let $a \in \mathbb{Q}^c$ be any irrational number.
Consider $A = (-\infty, a) \cap Y \neq \emptyset$ and $B = (a, \infty) \cap Y \neq \emptyset$
Also, $A \cap B = \emptyset$ and $A \cup B = Y$
Since, $(-\infty, a)$ and (a, ∞) are open in $X = \mathbb{R}$, then A, B are open in $Y = \mathbb{Q}$
Hence, A, B forms a separation of $\mathbb{Q}$ which means $\mathbb{Q}$ is not connected.

6. When we discuss about the topological space $\mathbb{R}$ we declare it as a connected space. But it should be noted that we only discuss about the usual topology on $\mathbb{R}$. Let us check if the topological space $\mathbb{R}$ is connected with respect to the lower limit topology.
In lower limit topology we have basis elements in $[a, b)$ form. So, to find a separation, let $B = [0, \infty)$ be an open set and let $A = (-\infty, 0)$.
Here, $A \neq \emptyset, \;\; B \neq \emptyset$.
$A \cap B = \emptyset, \;\; A \cup B = \mathbb{R}$
So, the only things require to A, B forms a separation of $\mathbb{R}$ is prove that $(-\infty, 0)$ is an open set.
We know that, $[-n, 0)$ is an open set in $\mathbb{R}_l$ for all $n \in \mathbb{R}$.
Thus $\cup_{n \in \mathbb{R}}[-n, 0) = (-\infty, 0)$ is an open set as union of arbitrary open sets is open.
Hence, A, B forms a separation of $\mathbb{R}_l$ which implies that $\mathbb{R}_l$ is not connected.

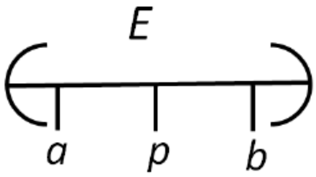
So, here we can see that connectedness of a topological space depends on the topology defined on it.

7. We already know about connectedness of a finite sequence. Now the question is how connectedness occurs in infinite sequence.
Suoppose, $\{A_\alpha\}$ be a collection of connected subspaces of X. Let A be another connected subspace of X. Then $A \cup (\cup A_\alpha)$ is connected if $A \cap A_\alpha \neq \emptyset \;\; \forall \alpha$ or $A \cap A_\alpha \neq \emptyset$ for any one α and the subspace $\cup A_\alpha$ is connected.

As we can see, the first condition is more general condition to prove. So, let us take the first condition and then prove it.
Let $A \cup (\cup A_\alpha) = B$. Now if possible let, C, D forms a separation of B. As A is connected, A lies either in C or in D.
Without loss of generality, let A lies in C ,i.e, $A \subseteq C$.
As, $A \cap A_\alpha \neq \emptyset \ \ \forall \alpha$ then $A_\alpha \subseteq C \ \ \forall \alpha$. Which further implies $D = \emptyset$.
We get a contradiction of our assumption.
Hence, $B = A \cup (\cup A_\alpha)$ is connected.

8. **Theorem:** Let E be a subset of the real line $\mathbb{R}$ containing at least two points. Then E is connected if and only if E is an interval.
Solution: suppose E not be and interval and contain atleast two point in $\mathbb{R}$

$$\exists a, b \in E \quad \text{and} \quad \exists p \notin E \quad \text{such that } a < p < b$$



Let $G = (-\infty, p)$ and $H = (p, \infty)$, then $a \in G$ and $b \in H$. So, $E \cap G$ and $E \cap H$ be two non-empty disjoint open sets whose union is E.
Therefore, G, H forms a separation of E.
Hence, if E is connected, it's an internal in $\mathbb{R}$.
Conversely, let E be an internal in $\mathbb{R}$. If possible, let E be disconnected. Then there exists a separation of E, say G, H forms a separation of E.
As, G, H is open in E, an interval in $\mathbb{R}$, then

$$G = (-\infty, p) \cap E \quad \text{and} \quad H = (p, \infty) \cap E \quad \text{for some} \quad p \in E$$

Then we will always found a

$$p \in E \quad \text{but} \quad p \notin G \cup H$$

So $G \cup H \neq E$, which contradicts that G, H forms a separation of E.
Therefore, E is connected.

We can prove the converse part in another words, i.e, let E be an interval in $\mathbb{R}$. Then E is homeomorphic to either of these following sets

$$(0,1) \quad \text{or} \quad [0,1] \quad \text{or} \quad (0,1] \quad \text{or} \quad [0,1)$$

As connectedness is a topological property, and all the above sets are connected, therefore E is connected.

9. From the definition of components, we know that each elements of a component have an equivalence relation between them.
It is also known from the set theory, that equivalence classes create a partition on a set. So we can claim that,
Components of X form a partition on X.

10. **Prove that,** if U be an open connected subspace of $\mathbb{R}^2$, then U is path connected.

Proof: Let $x \in U$ be a point and $A = \{y \in U : \ x \text{ and } y \text{ are path connected}\}$. So $A \subseteq U$ be a path connected subspace, as in $\mathbb{R}^2$ every subspace is locally path connected.
Also, as $x \in A$, A is non-empty.

If possible, let $A \neq U$. Since U is connected and A is open, $U \setminus A$ must also be open in U (as U being open in $\mathbb{R}^2$ makes A and $U \setminus A$ relatively open in U).This contradicts the connectedness of U because U would then be the union of two non-empty disjoint open sets A and $U \setminus A$.

Therefore, $A = U$, meaning every point in U can be connected to x by a continuous path.

11. The components of a totally disconnected space X are the singleton subsets of X. As,
Let E be a component of X and suppose

$$p, q \in E \quad \text{with} \quad p \neq q$$

Since X is totally disconnected, there exists a disconnection $G \cup H$ of X such that $p \in G$ and $q \in H$. Consequently, $E \cap G$ and $E \cap H$ are non-empty and so $G \cup H$ is a disconnection of E. But this contradicts the fact that E is a component and so is connected.
Hence E consists of exactly one point.

12. Let E be a component in a locally connected space X. Then E is open. As,
Let $p \in E$. Since X is locally connected, p belongs to at least one open connected set

G_p. But E is the component of p; hence

$$p \in G_p \subset E \quad \text{and so} \quad E = \bigcup \{G_p : p \in E\}$$

Therefore E is open, as it is the union of open sets.

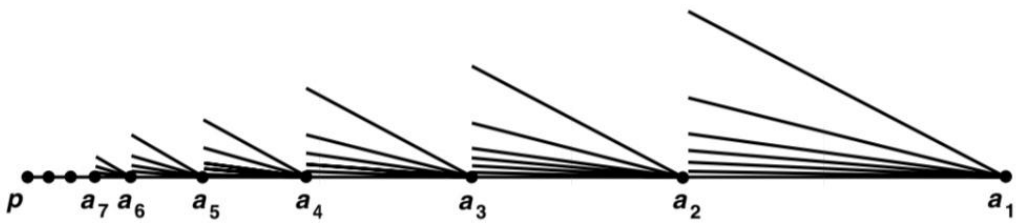
13. **Definition:** A space X is said to be **weakly locally connected at** x if for every neighborhood U of x, there is a connected subspace of X constrained in U that contains a neighborhood of x.

Now, we want to show that if X is weakly locally connected at each its points, then X is locally connected.

Here we show as the hint suggested that a component of an open subset of X is open. Let U be open in X and $C \subseteq U$ be its component. For any point $x \in C \subseteq U$ there is a connected subspace $S_x \subseteq X$ and an open neighborhood V_x such that $x \in V_x \subseteq S_x \subseteq U$. Since S_x is a connected subset of $U, S_x \subseteq C$. Therefore, C is the union of V_x for all $x \in C$ and is open. Another way to prove this is to show that the space is locally connected at every point. This way might be better considering the next exercise (in the proof we should somehow use the fact that the space is weakly locally connected at every point, or at least at every point in some neighrborhood of x, to prove that it is locally connected at x). Let U be a neighborhood of x. There is a connected subspace S such that it contains a neighborhood of x. Suppose there is a neighborhood of x such that the space is weakly locally connected at every point in the neighborhood. The intersection of these two neighborhoods of x is a neighborhood V of x such that it is contained in S and the space is weakly locally connected at every point in the neighborhood. If C is a component of V containing x then every point in C is contained in a connected subspace that a) has to be contained in C and b) contains a neighborhood of the point. Therefore, as before, we conclude that C is open in V, and, therefore, in X, contains x and is connected. I.e. the space is locally connected at x. Note that for this to be true we need the space to be weakly locally connected not only at the point x but at any point in some neighborhood of x.

The next theory shows that weak local connectedness at the point only does not imply the local connectedness at the point.

14. Consider the "infinite broom" X (shown in the below figure). X is not locally connected at p but weakly locally connected at p.


p
a7
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Applications of Connectedness

Connectedness is a fundamental concept in topology that has numerous applications across various fields of mathematics and science. Below is a detailed insight into the application of connectedness:

1. Analysis and Continuous Functions

- **Intermediate Value Theorem**: Connectedness is essential in the proof of the Intermediate Value Theorem, which states that if a continuous function f from a connected interval I to $\mathbb{R}$ takes values a and b at any two points in I, it also takes any value between a and b within I.
- **Connected Sets and Continuity**: The image of a connected set under a continuous function is connected. This property is used to analyze the behavior of continuous functions, especially in complex analysis and real analysis.

2. Topology and Geometry

- **Path-Connected Spaces**: Connectedness helps in classifying topological spaces. Path-connected spaces, where any two points can be joined by a continuous path, simplify the study of topological properties.
- **Separation and Compactness**: Connectedness is used in conjunction with compactness to understand the structure of topological spaces. For example, the fact that the continuous image of a compact connected space is compact and connected helps in various geometric proofs.

3. Complex Analysis

- **Analytic Continuation**: In complex analysis, connectedness is crucial for the concept of analytic continuation, where a holomorphic function defined on a connected open subset can be extended to a larger domain.
- **Riemann Surfaces**: The study of Riemann surfaces, which are connected complex manifolds, relies heavily on connectedness to ensure the surface behaves as a single entity.

4. Dynamical Systems

- **Phase Space Analysis**: In the study of dynamical systems, connectedness of phase space components helps in understanding the behavior of trajectories over time. Stable and unstable manifolds are typically connected.

- **Invariant Sets**: The concept of connectedness is used to study invariant sets under dynamical systems, which are crucial for understanding long-term behavior and stability.

5. **Algebraic Topology**

- **Homotopy and Fundamental Group**: Connectedness is foundational in defining the fundamental group, which measures the loops in a space. Path-connected spaces ensure that the fundamental group is well-defined.
- **Covering Spaces**: Connectedness plays a role in the theory of covering spaces, where the properties of a covering space are deeply linked to the connectedness of the base space.

6. **Mathematical Physics**

- **Quantum Mechanics**: In quantum mechanics, the connectedness of configuration spaces ensures the proper definition of wave functions and their evolution.
- **Relativity**: In general relativity, the connectedness of spacetime ensures that the fabric of space and time is continuous, which is crucial for the definition of causal relationships and the propagation of signals.

7. **Differential Equations**

- **Existence of Solutions**: Connectedness is used in proving the existence of solutions to differential equations. For example, solutions to certain differential equations may be connected curves in the solution space.
- **Boundary Value Problems**: Connectedness helps in understanding the structure of solution sets for boundary value problems, ensuring that solutions form continuous families.

8. **Graph Theory**

- **Connected Graphs**: In graph theory, connectedness ensures that there is a path between any two vertices in a graph, which is essential for network design, communication, and transportation problems.
- **Component Analysis**: The study of connected components in a graph helps in clustering, finding communities in social networks, and analyzing molecular structures in chemistry.

9. Data Science and Machine Learning

- **Clustering Algorithms**: Connectedness is a key concept in clustering algorithms like DBSCAN (Density-Based Spatial Clustering of Applications with Noise), where clusters are formed based on the density and connectivity of points.
- **Topological Data Analysis**: Connectedness is used in persistent homology, a method in topological data analysis that studies the shape of data and its features at different scales.

Conclusion

Connectedness is a versatile and powerful concept in topology with wide-ranging applications in various branches of mathematics and science. Its role in ensuring the integrity and continuity of spaces makes it a fundamental property used to solve complex problems and understand intricate structures.

5 Compactness:

The notion of compactness is not nearly so natural as that of connectedness. From the beginnings of topology, it was clear that the closed interval $[a, b]$ of the real line had a certain property that was crucial for proving such theorems as the maximum value theorem and the uniform continuity theorem. But for a long time, it was not clear how this property should be formulated for an arbitrary topological space. It used to be thought that the crucial property of $[a, b]$ was the fact that every infinite subset of $[a, b]$ has a limit point, and this property was the one dignified with the name of compactness. Later, mathematicians realized that this formulation does not lie at the heart of the matter, but rather that a stronger formulation, in terms of open coverings of the space, is more central. The latter formulation is what we now call compactness.
It is not as natural or intuitive as the former; some familiarity with it is needed before its usefulness becomes apparent.

Definition (cover and open cover) : A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition (compact space) : A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Let's check some popular topological spaces whether they are compact or not.

EXAMPLE 1 The real line $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ by open intervals

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection that covers $\mathbb{R}$.

EXAMPLE 2 The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}.$$

Given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing 0 . The set U contains all but finitely many of the points $1/n$; choose, for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite subcollection of $\mathcal{A}$ that covers X.

EXAMPLE 3 Any space X containing only finitely many points is necessarily compact, because in this case every open covering of X is finite.

EXAMPLE 4 The interval $(0, 1]$ is not compact; the open covering

$$\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{Z}_+\}$$

contains no finite subcollection covering $(0, 1]$. Nor is the interval $(0, 1)$ compact; the same argument applies. On the other hand, the interval $[0, 1]$ is compact; you are probably already familiar with this fact from analysis. In any case, we shall prove it shortly.

In general, it takes some effort to decide whether a given space is compact or not. First we shall prove some general theorems that show us how to construct new compact spaces out of existing ones. Then in the next section we shall show certain specific spaces are compact. These spaces include all closed intervals in the real line, and all closed and bounded subsets of $\mathbb{R}^n$.

Let us first prove some facts about subspaces. If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to cover Y if the union of its elements contains Y.

Lemma Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.
Proof Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{A_{\alpha_1} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a subcollection of $\mathcal{A}$ that covers Y.
Conversely, suppose the given condition holds;we wish to prove Y compact.
Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite subcollection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers Y. Then $\left\{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\right\}$ is a subcollection of $\mathcal{A}'$ that covers Y.
Hence, the statement is proved.

Theorem Every closed subspace of a compact space is compact.
Proof: Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X.
Let us form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X - Y$, that is,

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}.$$

Now, Some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X - Y$, we will discard $X - Y$.
Otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y.
Which completes the proof.

Theorem Every compact subspace of a Hausdorff space is closed.

Proof: Let Y be a compact subspace of the Hausdorff space X. We shall prove that $X - Y$ is open, so that Y is closed.

Let x_0 be a point of $X - Y$. We show there is a neighborhood of x_0 that is disjoint from Y.

Now, for each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (using the Hausdorff condition). The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X; therefore, finitely many of them $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ cover Y. The open set

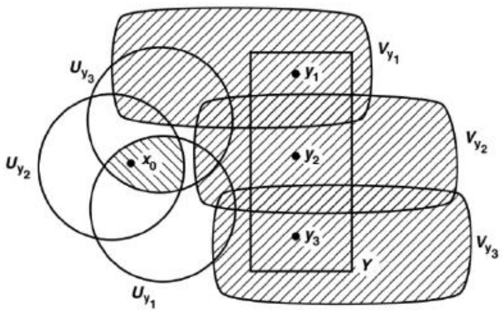
$$V = V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}$$

contains Y, and it is disjoint from the open set

$$U = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$$

formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{y_i}$ for some i, hence $z \notin U_{y_i}$ and so $z \notin U$. (described in the below figure)

Then U is a neighborhood of x_0 disjoint from Y, as desired.



In the proof of the above theorem, we proved another very important statement which will be further required. We are mentioning this statement for later uses.

Lemma: If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively

From the previous two theorems, we get that every closed subspace of a compact space is compact and every compact subspace of a compact space is closed, but for the second condition, Hausdorff property is required.

Which means there is some examples of compact spaces which is not Hausdorff and whose compact subspaces are not closed.

e.g, Let us take the subspace as Co-finite topology on $\mathbb{R}$. Here, every subspace of $\mathbb{R}$ is compact but only closed sets in this topology are the finite sets.

Also in this space, the Hausdorff property is not satisfied. Hence, for the second theorem, Hausdorff property is required.

theorem The image of a compact space under a continuous map is compact.

proof: Let $f : X \to Y$ be continuous; let X be compact. Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them, say

$$f^{-1}(A_1), f^{-1}(A_2) \dots, f^{-1}(A_n),$$

cover X, as X is compact. Then the sets $A_1, A_2, \dots, A_n$ cover $f(X)$.

theorem Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

proof: To prove that f is a homeorphism, we need to show that f^{-1} is continuous. For that, we shall prove that images of closed sets of X under f are closed in Y.

Let A be any closed subspace of X, which is a compact space. Then A is a compact subspace of X.

Now, as f is continuous so $f(A)$ is also compact subspace of Y, as under continuous map image of compact subspace is compact.

Since, Y is a Hausdorff space and $f(A)$ is a compact subspace of Y, so $f(A)$ is closed in Y. Hence f^{-1} is continuous.

Therefore, f is a homeomorphism. Which completes the proof.

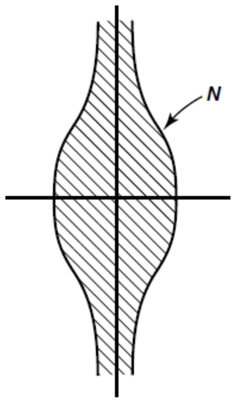
Now, like connectedness we should discuss compctness on the product topological spaces too.

Theorem: The product of finitely many compact spaces is compact.

There is nothing more specefic about proof of this theorem. But here comes an important concept about tubes.

The tube lemma: Consider the product space $X \times Y$, where Y is compact. If N be an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighbourhood of x_0 in X.

The image given below explains a tube graphically.



Now, in the above theorem we talked about finite product of topological spaces. So now the question is what about product of infinitely many compact topological spaces? Is it compact or not?

The answer is "yes". Product of infinitely many compact spaces is compact under the product topology and we find it from the **Tychonoff theorem.**

5.1 Compactness on $\mathbb{R}$

At the introduction part of this section we discussed about the topic of compactness first shown in the real analysis. In studies of limits points, continuity, convergence of sequences there was something important in the closed intervals $[a, b]$. So now we will again come to the real numbers and will imply the theories of compactness on $\mathbb{R}$. Here, we will prove those concepts of compactness with the examples of real numbers.

So, at first we will start with some theorems.

Theorem: Let X be a simply ordered set having the least upper bound property. In the ordered topology, each closed interval in X is compact.

Proof: Given that X is an ordered set. Then choose $a < b$, Let $\mathcal{A}$ be an open covering of $[a, b]$ by sets open in $[a, b]$ by subspace topology (Here it is ordered topology). We want to prove that there exists a finite subcollection of $\mathcal{A}$ which covers $[a, b]$.

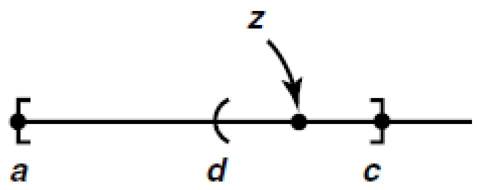
Now, if we take any point $x \in [a, b]$ such that $x \neq b$, then there exists a point y in $[a, b]$ with $x < y$. We will show that the interval $[x, y]$ can be covered by atmost two elements of $\mathcal{A}$.

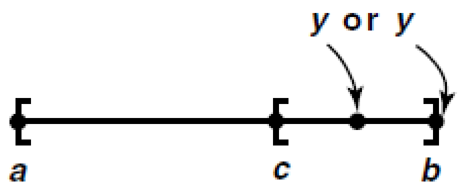
Case 1 If x has a immediate successor in X, let y be that immediate successor of x and hence $[x, y]$ consists of only two points x and y. So, this can be easlily covered by two elements of $\mathcal{A}$.

Case 2 If x has no immediate successor in X, then we choose an element A of $\mathcal{A}$ which contains x. As $x \neq b$ and A is an open set in X, then A contains an interval of the form $[x, c)$ for some $c \in [a, b]$.
We will choose a point $y \in [x, c)$, then we find an interval $[x, y]$ which is covered by a single element A of $\mathcal{A}$.

Now, let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by a finite subcollection of $\mathcal{A}$. Here, C is non-empty, so applying least upper bound property, there exist a least upper bound c of the set C such that $a < c \leq b$.

Now, the only remaining proof is $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Let us choose an element A from $\mathcal{A}$ which contains c. Since A is open it contains an interval $(d, c]$ for some $d \in [a, b]$.
If c is not in C, then there exists a point z of C lying in the interval (d, c), as otherwise d would be smaller upper bound of C than c (The scenario is given in the below figure).
Since $z \in C$ the interval $[a, z]$ can be covered by finitely many elements of $\mathcal{A}$, say n number of elements. Also $[z, c]$ lies in a single element A of $\mathcal{A}$.
Hence, $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. Thus $c \in C$.





Finally, we claim that $c = b$.
If possible, let $c < b$. Then we take $x = c$ and therefore there exists a point $y > c$ of $[a, b]$ such that the interval $[c, y]$ can be covered by finite elements of $\mathcal{A}$. Also we proved that $[a, c]$ can be covered by finite elements of $\mathcal{A}$. Therefore the interval

$$[a, y] = [a, c] \cup [c, y]$$

can also be covered by finitely many elements of $\mathcal{A}$, implies that $y \in C$.
This is a contradiction of the fact that c is an upper bound of C.
Hence, $c = b$.

So, we get that the interval $[a, b]$ can be covered by a finite number of elements of $\mathcal{A}$. Therefore $[a, b]$ is compact.

As we all know the set of real numbers $\mathbb{R}$ is an ordered set having least upper bound property, so this theorem applies on $\mathbb{R}$. i,e,

Corollary: Every closed interval in $\mathbb{R}$ is compact.

For more genereal approach if we consider the space $\mathbb{R}^n$ we will not have a proper upper bound or boundedness on it. So we will take the help of metric for these issues. It is previously mentioned that every metric spaces is also a topological space. So, the theorem goes like,

Theorem: A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.

Extreme value theorem: Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist two points $c, d \in X$ such that

$$f(c) \leq f(x) \leq f(d) \quad \text{for every} \quad x \in X$$

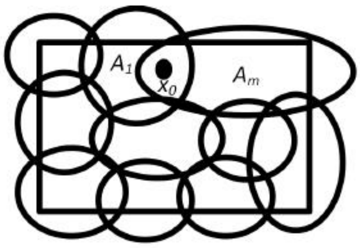
The lebesgue number lemma: Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there is an element of $\mathcal{A}$ containing it.

The number δ is called a **Lebesgue number** for the open covering $\mathcal{A}$.

Proof: Given, $\mathcal{A}$ is an open covering of X.
If X itself is an element of $\mathcal{A}$, then any positive number is a lebesgue number for $\mathcal{A}$.

Let us assume that X is not an element of $\mathcal{A}$.
Let us choose a finite subcollection $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$.
Let $C_i = X - A_i$ for all $i = 1, 2, \cdots, n$ and let us define a function $f : X \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$



Now, we will show that, $f(x) > 0 \quad \forall\, x \in X$.
Let $x \in X$, then $x \in A_i$ for some $i \in \{1, 2, \cdots, n\}$. Let us choose a $\epsilon > 0$ such that ϵ- neighbourhood of x lies in A_i.

$$\text{Therefore} \quad f(x) \geq \frac{\epsilon}{n}$$

$$\text{Hence,} f(x) > 0$$

Since f is continuous an X is compact, then it has a minimum value, say δ. We claim that this δ is our required lebesgue number.
Let B is a subset of X and $x_0 \in B$. Also let diameter of B is less than δ, then B lies in a δ- neighbourhood of x_0.

$$\text{Now,} \delta \leq f(x_0) \leq d(x_0, C_m)$$

Where $d(x_0, C_m)$ is the largest element of $d(x_0, C_i) \ \forall\, i$.

Then δ- neighbourhood of x_0 is contained in the element $A_m = X - C_m$ of the covering $\mathcal{A}$.

Hence, the lemma got proved.

For checking the compactness in $\mathbb{R}$, we just have one last theorem on countability.

Theorem: Let X be a non-empty compact Hausdorff space. If X has no isolated point, then X is uncountable.

With the help of this theorem, we get that

Corollary: Every closed interval in $\mathbb{R}$ is uncountable.

5.2 Local compactness:

Here we will discuss about local compactness. This area is also important because here we will learn to the topic of compactification.

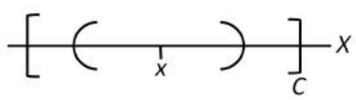
This is a common topic that the set $(a, b]$ is not compact but by just adding a point a makes the set

$$\{a\} \cup (a, b] = [a, b]$$

which is a compact set.
So there is some genereal theory not just for $\mathbb{R}$ but also for other spaces. At first we need to understand the topic of locally compactness which is kind of similar to the topic of locally connectedness. Let's start with the defintion.

Definition: A space X is said to be locally compact at a point x if there is some compact subspace C of X that contains a neighbourhood of x.

If X is locally compact at each of its points, then X is called locally compact.



By the defintion, it is clear that any compact space is always locally compact. Let us see some examples whether they are locally compact spaces or not.

Example- 1 The real line $\mathbb{R}$ is a locally compact space. As if we take a point x which lies in the interval (a, b) then there exists a compact subspace $[a, b]$ of $\mathbb{R}$ which contains the neighbourhood (a, b).

Example- 2 The set of all rational numbers $\mathbb{Q}$ is not locally compact. As for $x \in \mathbb{Q}$, there exists a neighbourhood $(a, b) \cap \mathbb{Q}$ of x.
Let $q_1, q_2, \cdots \in (a, b) \cap \mathbb{Q}$ and consider the collection,

$$\mathcal{E} = \{A_i \quad \text{open in} \quad \mathbb{Q} : A_i \quad \text{contains only} \quad q_i \;\; \forall\, i \in \Lambda\}$$

Then $\mathcal{E}$ is an open cover of the set $(a, b) \cap \mathbb{Q}$ and $\mathcal{E}$ has no finite subcollection.

Therefore, any superset of $(a, b) \cap \mathbb{Q}$ in $\mathbb{Q}$ is not compact.

Theorem: Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following condition.
(i) X is a subspace of Y.
(ii) The set $Y - X$ consists of a single point.
(iii) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals to the identity map on X.

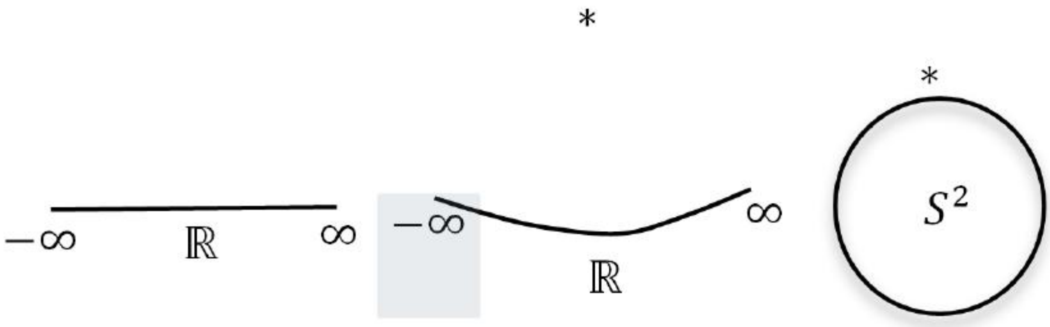
The statement of the above theorem is very important as it gives us further knowledge of the topic compactification.

Compactification: If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals to Y, then Y is said to be a compactification of X.
If $Y - X$ equals a single point, then Y is called the one-point compactification of X.

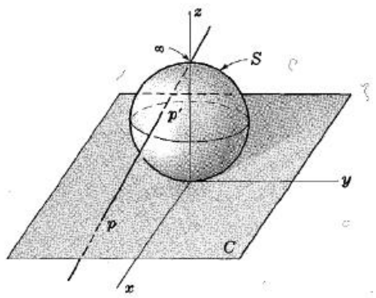
Let's take a view on some examples of compactification.

Examples:
(i) The compactification of the space $(0, 1)$ or $(0, 1]$ or $[0, 1)$ are the compact space $[0, 1]$ with respect to the usual topology.

(ii) The set of all real numbers $\mathbb{R}$ is a locally compact space but it is not compact. So, we can use compactification on $\mathbb{R}$.
From the real line we say there is some $-\infty, \infty$. If we add these two points into a point(like geting a circle from a wire), then we find a circle S which is a compactification of $\mathbb{R}$.
The process is given in the below figure aboout how the point $*$ is created from $-\infty$ and ∞.



(iii) By the exactly same argument from the example-(ii), it is very clear that the one point compactification of the set $\mathbb{R}^2$ is homeomorphic to a sphere, say S^2. (Here, we take an infinite number of points to a single point, say $*$). The graphical representation is given in the below figure.



There is another formulation of local compactness. This is equivalent to the definition of the local compactness when the space is Hausdorff. For the general study, let's start this formulation as a theorem.

Theorem: Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof: Suppose for $x \in X$ and a neighbourhood U of x, there exists a neighbourhood V of X such that $\overline{V}$ is compact and $\overline{V} \subset U$.
Then for $x \in V$ (a neighbourhood of x) there exists a compact subspace $\overline{V}$ such that

$$x \in V \subseteq \overline{V}$$

Hence, X is locally compact at x.
Since x is arbitrary, X is locally compact.

Conversely, Suppose, X is locally compact.
Let $x \in X$ and U be a neighbourhood of x.

As, X is a local compact space, there is an one point compactification Y of X, which is compact.
Let $C = Y - U$, then C is closed in Y. Since, Y is compact and C is closed in Y, so C is also compact. Then there exists two disjoint open sets V and W containing x and C respectively
Here we find a V such that $\overline{V}$ is compact in Y, as $\overline{V}$ is closed in the compact space Y.
Also $\overline{V}$ is disjoint from C.

$$\text{i.e,} \quad \overline{V} \cap C = \emptyset$$

$$\text{Hence,} \quad \overline{V} \subset U$$

Which completes the proof.

Corollary: Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proof: Given that X is a locally compact Hausdorff space and A is a subspace of X.

Suppose that A is closed in X. Let $x \in A$. As, X is locally compact and $x \in X$, there exists a compact subspace C of X which contains a neighbourhood U of x.
As, C is compact and A is a closed subspace of X, then $C \cap A$ is closed in C and hence compact. Also $C \cap A$ contains the neighbourhood $A \cap U$ (open in A) of x in A.
As x is arbitrary, A is locally compact.

Suppose that A is open in X. Let $X \in A$.
As, X is a Hausdorff space, which is locally compact, using the above theorem we get a V of X such that $\overline{V} \subset A$ and $\overline{V}$ is compact.
Then we find a $C = \overline{V}$ a compact subspace of A, which contains the neighbourhood V of X in A.
As x is arbitrary, A is locally compact.

In the proof of the above corollary, if A is a closed space then the theorem is not required. That means if X is not a Hausdorff space, A will be locally compact if A is closed.
But for an open subspace A the Hausdorff property of X should be required.

Corollary: A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

Definition:(Equi-continuous): Let $\mathcal{F}$ be a collection of real valued continuous functions in a metric space (X, d). Then $\mathcal{F}$ is called equi-continuous if for given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$, then

$$| f(x) - f(y) | < \epsilon \quad \forall f \in \mathcal{F}$$

Before ending this part we should add one more theorem.

The Arzela-Ascoli Theorem: Let X be a compact metric space and $\mathcal{F}$ be a subset of $c(X)$, then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed, uniformly bounded and equi-continuous.

5.3 More theories on compactness:

In this section, we are going to discuss more topics on Compactness which will help to understand some basic problems with more clear view and will also discuss some advance topics.

So lets start them, one by one.

1. In some theories discussed before, we claimed that a topological space with co-finite topology is compact. Here, we will prove that.
Let $(X, \mathcal{T})$ is a topological space, where $\mathcal{T}$ is the co-finite topology on the set X.
Let $\mathcal{A}$ be an open cover of the space X and let $A \in \mathcal{A}$, then A is an open set of X. Therefore, A^c contains finite number of elements, where, A^c means complement of A. Let A^c contains n points of X, i.e,

$$\text{let} \quad A^c = \{a_1, a_2, \cdots, a_n\}$$

As, $\mathcal{A}$ is an open cover of X, then for each element of X, there exists an element of $\mathcal{A}$ which contains the element of X.

$$\text{i.e,} \quad \forall \;\; a_i \in A^c \quad \exists \quad A_i \in \mathcal{A} \quad \text{such that} \quad a_i \in A_i \quad \forall i = 1, 2, \cdots n$$

Then $A^c \subseteq \bigcup_{i=1}^{n} A_i \quad \forall \;\; i = 1, 2, \cdots, n.$
As. $X = A \cup A^c$, then

$$X = A \cup A_1 \cup A_2 \cup \cdots \cup A_n$$

So, we find a finite sub-collection of $\mathcal{A}$ which covers X. Hence X is compact

2. Theorem: Let A be a subset of the topological space $(X, \mathcal{T})$. Then the following conditions are equivalent.
(i) A is compact with respect to $\mathcal{T}$
(ii) A is compaft with respect to the relative topology $\mathcal{T}_A$ on A.

Proof: At first, we are going to show that $(i) \implies (ii)$.
i.e, A is compact with respect to $\mathcal{T}$.
Let $\mathcal{A}$ is a $\mathcal{T}_A$ open cover of A. Then by the defintion of relative topology,

$$\forall A_i \in \mathcal{A}, \quad \exists \; B_i \in \mathcal{T} \quad \text{such that} \quad A_i = A \cap B_i \subset B_i$$

Hence,

$$A \subset \cup_i A_i \subset \cup_i B_i$$

So, we can find a set $\mathcal{B}$ consists of those B_i which is a $\mathcal{T}$ open cover of A.
Now, as A is compact with respect to $\mathcal{T}$ then there exists a finite subcollection of $\mathcal{B}$ which covers A.
Let $\{B_1, B_2, \cdots, B_n\}$ be a finite sub-collection of $\mathcal{B}$ which covers A. Then

$$\forall B_i \in \mathcal{B} \quad \exists A_i \in \mathcal{A} \quad \forall i = 1, 2, \cdots, n$$

which covers A. It is because of the formation of all the B_i.
Hence, A is compact with respect to $\mathcal{T}_A$.

Now, we will show that $(ii) \implies (i)$
Let A is a compact space with respect to the relative topology $\mathcal{T}$. Let $\mathcal{B}$ is a $\mathcal{T}$ open cover of A. Then by the defintion of relative topology,

$$\forall \ B_i \in \mathcal{B} \quad \exists \ A_i \in \mathcal{T}_A \quad \text{such that} \quad A \cap B_i = A_i$$

Taking these A_i, we make a collection $\mathcal{A}$ which is an open cover of A with respect to $\mathcal{T}_A$.
Now, A is compact with respect to $\mathcal{T}_A$. So, there exists a finite subcollection of $\mathcal{A}$ which covers A. Let $\{A_1, A_2, \cdots, A_m\}$ covers A.
Then in the similar manner, we get $\{B_1, B_2, \cdots, B_m\}$, a finite sub-collection of $\mathcal{B}$ which covers A.
Hence, A is also compact with respect to the topology $\mathcal{T}$.

3. From the previous theorem, we find a preceeding corollary,
Corollary: Let $(Y, \mathcal{T}')$ be a subspace of $(X, \mathcal{T})$ ans let $A \subset Y \subset X$. Then A is $\mathcal{T}$-compact if and only if A is $\mathcal{T}'$-compact.

4. There is another theorem on the basic of compactness. It can used as an alternative defintion of compactness. As here we will consider the complements of the elements of the open cover, that is some closed sets.
Theorem: Let $(X, \mathcal{T})$ is a topological space. Then the following statements are equivalent.
(i) X is compact.
(ii) For every class $\{F_i\}$ of closed subsets of X, $\cap_i F_i = \emptyset$ implies $\{F_i\}$ contains a finite subcollection $\{F_1, F_2, \cdots, F_n\}$ with $F_1 \cap F_2 \cap \cdots \cap F_n = \emptyset$.

Proof: First, we will prove that $(i) \implies (ii)$
Then X is a compact space. Let $\cap_i F_i = \emptyset$. Then by De-Morgan's law,

$$X = \emptyset^c = (\cap_i F_i)^c = \cup_i F_i^c$$

As, all F_i is closed F_i^c is open and therefore $\{F_i^c\}$ is an open cover of X.
Now, as X is compact, then there exists a finite sub-collection $\{F_1^c, F_2^c, \cdots, F_n^c\}$ of $\{F_i^c\}$ which covers X. So,

$$F_1^c \cup F_2^c \cup \cdots \cup F_n^c = X$$

Again using De-Morgan's law,

$$\begin{aligned} F_1 \cap F_2 \cap \cdots F_n &= (F_1^c \cup F_2^c \cup \cdots \cup F_n^c)^c \\ &= X^c \\ &= \emptyset \end{aligned}$$

Which completes the proof.
Now, we will show that $(ii) \implies (i)$
Let the condition (ii) holds. We wish to show that X is compact. For that, Let $\mathcal{A}$ is an open covering of X, we need to find a finite sub-collection of $\mathcal{A}$ which covers X. As all the elements of $\mathcal{A}$ is open then complements of these sets is closed. Then we find a collection $\{A_i^c\}$ of closed sets in X such that

$$\cup_i A_i = X \implies \cap_i A_i^c = \emptyset \quad \text{Using, De-Morgan's law}$$

Then using the given condition we find a finite sub-collection $\{A_1^c, A_2^c, \cdots, A_n^c\}$ such that $A_1^c \cap A_2^c \cap \cdots \cap A_n^c = \emptyset$
Again, using De-Morgan's law

$$\begin{aligned} A_1 \cup A_2 \cup \cdots A_n &= (A_1^c \cap A_2^c \cap \cdots A_n^c)^c \\ &= \emptyset^c \\ &= X \end{aligned}$$

Then we find a finite subcollection $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$ which covers X.
Hence, X is compact.

5. In the previous discussion it was shown that there is some relations of compactness with the Hausdorff property. There was a lot of theory we discussed of compactness where the Hausdorff property was satisfied. To find a relation among them there should be an similarity of them. So the next theorem will provide that.
Theorem: Let f a one-one continuous function from a compact space X into a Hausdorff space Y. Then X and $f(X)$ are homeomorphic.

Proof: Given $f : X \to Y$ is a one-one continuous function. Let $f : X \to f(X)$ be a restricted map of f. So, g is a one-one,onto and a continuous map.
Hence, g^{-1} exists. We have just one thing to prove and that is $g^{-1} : f(X) \to X$ is continuous.
Let F is a closed subset of X, we whish to show that

$$(g^{-1})^{-1}(F) = g(F)$$

is closed in $g(X)$.
As, F is a closed subspace of a compact space X, then F is compact.

As g is continuous and F is compact in X, $g(F)$ is compact in $f(X)$.
As, $f(X)$ satisfies Hausdorff property and $g(F)$ is closed in $f(X)$, $g(F)$ is closed in $f(X)$.
Hence, g^{-1} is a continuous mapping.

$$\therefore g : X \to f(X) \quad \text{is a homeomorphism}$$

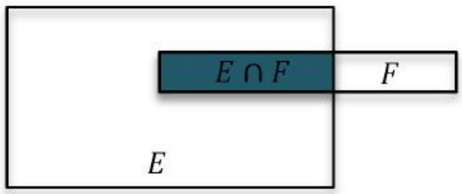
Hence,

$$X \simeq f(X)$$

6. Corollary: Let $(X, \mathcal{T})$ be a compact and let $(X, \mathcal{T}')$ be a Hausdorff space. Then $\mathcal{T}' \subseteq \mathcal{T}$ implies $\mathcal{T}' = \mathcal{T}$.

7. If E is compact and F is closed, then $E \cap F$ is compact.

The proof of this theorem is an obvious concept. We will understand this with help of diagram.



Given E is compact and F is closed. Then $E \cap F$ is closed in E.
As, E is compact and $E \cap F$ is closed in E, therefore $E \cap F$ is compact in E, i.e, $E \cap F$ is compact.

Applications of Compactness

Compactness is a key concept in topology and analysis, playing a vital role in various branches of mathematics and its applications. Below is a detailed insight into the applications of compactness:

1. Analysis

- **Extreme Value Theorem**: In real analysis, the Extreme Value Theorem states that a continuous function on a compact set attains its maximum and minimum values. This is fundamental in optimization problems and in ensuring the existence of extremal values.
- **Uniform Continuity**: Continuous functions on compact sets are uniformly continuous. This property is crucial in approximation theory and in the study of differential equations, where it guarantees that functions behave well across the entire domain.

2. Topology

- **Heine-Borel Theorem**: In $\mathbb{R}^n$, the Heine-Borel Theorem characterizes compact sets as those that are closed and bounded. This theorem is fundamental in understanding the structure of subsets in Euclidean space.
- **Compactness in Metric Spaces**: In metric spaces, compactness is used to ensure that every sequence has a convergent subsequence (Bolzano-Weierstrass property), which is important in convergence analysis and functional analysis.

3. Functional Analysis

- **Banach-Alaoglu Theorem**: In functional analysis, the Banach-Alaoglu Theorem states that the closed unit ball in the dual space of a normed vector space is compact in the weak* topology. This result is crucial in the study of dual spaces and weak convergence.
- **Riesz Representation Theorem**: This theorem uses compactness to represent continuous linear functionals on Hilbert spaces as inner products, facilitating the analysis and solution of various problems in Hilbert spaces.

4. Measure Theory and Integration

- **Compact Support**: Functions with compact support (i.e., functions that are zero outside a compact set) are important in measure theory and integration, particularly in defining and working with distributions and in the theory of Sobolev spaces.

- **Lebesgue Dominated Convergence Theorem**: Compactness is used to ensure conditions for the Lebesgue Dominated Convergence Theorem, which allows the interchange of limit and integral operations under certain conditions.

5. **Differential Equations**

- **Existence of Solutions**: Compactness is used in proving the existence of solutions to differential equations, particularly through the Arzelà-Ascoli Theorem, which provides criteria for precompactness of families of functions.

- **Boundary Value Problems**: In solving boundary value problems, compactness arguments ensure that certain operators are well-behaved and that solutions exist within a specified function space.

6. **Dynamical Systems**

- **Invariant Sets**: In the study of dynamical systems, compact invariant sets ensure that trajectories do not escape to infinity and help in the analysis of long-term behavior and stability of the system.

- **Poincaré Recurrence Theorem**: This theorem uses compactness to state that certain systems will return arbitrarily close to their initial states after some time, which is significant in the study of ergodic theory and statistical mechanics.

7. **Algebraic Topology**

- **Homology and Cohomology**: Compactness simplifies the computation of homology and cohomology groups, which are used to classify topological spaces and understand their properties.

- **Compactly Generated Spaces**: In algebraic topology, working with compactly generated spaces helps in simplifying the theory and ensuring that certain constructions and results are well-defined and manageable.

8. **Mathematical Physics**

- **Quantum Mechanics**: In quantum mechanics, the compactness of certain operators (such as the Hamiltonian in a bounded potential) ensures the discreteness of the spectrum, which corresponds to quantized energy levels.

- **Relativity**: Compactness is used in general relativity to analyze the global structure of spacetime, ensuring that certain properties hold over the entire manifold.

9. Optimization and Numerical Analysis

- **Optimization Problems**: Compactness ensures that optimization problems have solutions within a feasible region, as in the case of linear programming problems where feasible regions are often compact polyhedra.
- **Convergence of Algorithms**: In numerical analysis, compactness is used to prove the convergence of iterative algorithms, ensuring that sequences generated by the algorithms have accumulation points that are potential solutions.

10. Economics and Game Theory

- **Existence of Equilibria**: Compactness is crucial in proving the existence of equilibria in economic models and game theory, such as Nash equilibria, by ensuring that certain sets are closed and bounded.
- **Fixed Point Theorems**: Many fixed point theorems, which are fundamental in economics for proving the existence of solutions to equilibrium problems, rely on compactness. For example, the Brouwer and Kakutani Fixed Point Theorems require compactness assumptions.

Conclusion

Compactness is a powerful and versatile concept with widespread applications in mathematics and science. It provides a framework for ensuring the existence and properties of solutions in various problems, from analysis and topology to differential equations and mathematical physics. Its role in guaranteeing the boundedness and continuity of spaces makes it an indispensable tool in both theoretical and applied mathematics.