SOME COMMON FIXED POINT THEOREMS FOR CONTRACTING MAPPINGS IN BICOMPLEX VALUED b-METRIC SPACES

Article · January 2020

4 authors:


Sanjib Kumar Datta
University of Kalyani


Dipankar Pal
Prof. Syed Nurul Hasan College


Rakesh Sarkar
Gour Mahavidyalaya


Jayanta Saha
UNIVERSITY OF KALYANI, P.O.: KALYANI, DIST: NADIA, PIN:741235, WEST BENGAL.

Bull. Cal. Math. Soc., **112,** (4) 329–354 (2020)

SOME COMMON FIXED POINT THEOREMS FOR CONTRACTING MAPPINGS IN BICOMPLEX VALUED b-METRIC SPACES

SANJIB KUMAR DATTA[1], DIPANKAR PAL[2],
RAKESH SARKAR[3] AND JAYANTA SAHA[4]

(*Received 27 May 2020 and revision received 9 August 2020*)

Abstract. In this paper we define the bicomplex valued b-metric space and study some common fixed point theorems in bicomplex valued b-metric spaces satisfying some rational inequality for a pair of self contracting mappings.

AMS Subject Classification (2010) : 30G35, 46N99.

Keywords and phrases : Common fixed point, Contractive type mapping, Bicomplex valued metric space, Bicomplex valued b-metric space.

I. Introduction, Definitions and Notations. Segre's (1892) paper, published in 1892 made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this significant work of Segre failed to earn the attention of the mathematicians for almost a century. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology.

Price (1991) developed the bicomplex algebra and function theory. In this field an impressive body of work has been developed by different researchers during the last few years. One can see some of the attempts in (Agarwal et al., 2014, 2015, 2014, Banerjee et al., 2014, Charak et al., 2013, Luna-Elizarrarás et al., 2012, Alpay et al., 2014, Luna-Elizarrarás et al., 2013, Lavoie et al., 2010, 2011, 2011, Rochon, 2004, Kumar and Kumar 2011, Rochon and Shapiro, 2004, Spampinato, 1935, 1936, Scorza Dragoni, 1934, Sintunavarat, 2011, De et al., 2012, Colombo et al., 2010, Goyal, 2007, Jaishree, 2012, and Kumar and Dixit, 2011).

In 2011, Azam et al. (2011) introduced a concept of complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard et al. (2012) generalized the result obtained by Azam et al. (2011) and proved another common fixed point theorem satisfying some rational inequality in complex valued metric space. In fact, the fixed point theory has been studied in different types of metric spaces. The significant contributions of Tripathy et al. (2013, 2014) should be mentioned here. The main tool which is used to prove the fixed point theorems is the Banach contraction principle and it states: if (X, d) be a complete metric space and $T : X \to X$ is a self-map then $d(Tx, Ty) \leq ad(x, y)$ where $0 \leq a < 1$, then T has a unique fixed point. Banach proved this theory in 1922. In this connection Choudhury et al. (2015) proved some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Datta and Ali (2017) proved common fixed point theorems for four mappings in complex valued metric spaces. Also one can see the attempts in (Ahmad et al., 2014, Bhatt et al., 2011, Tiwari and Shukla, 2012, Verma and Pathak, 2013 and Bhatt, Chaukiyal and Dimri, 2011).

The concept of complex-valued b-metric spaces is introduced by Rao et al. (2013) in 2013 and they prove a common fixed point theorem in complex valued b-metric spaces. In his article Mukheimer (2014) proved some common fixed point theorems in complex-valued b-metric spaces. Also Dubey et al. (2015) proved some common fixed point theorems for contractive mappings in complex-valued b-metric spaces.

Recently, Choi et al. (2017) introduced the concept of bicomplex valued metric spaces and proved some common fixed point theorems in connection with two weakly compatible mappings. Jebril et al. (2019) prove some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. Subsequently, Datta et al. (2020) studied some more results regarding the generalization of fixed point theorems in bicomplex valued metric spaces. The attempt made by Datta and Pal (2020) can also be regarded as a recent contribution in this field.

In this paper going to study some common fixed point theorems for contractive mappings in bicomplex valued b-metric spaces.

Let $z_1, z_2 \in \mathbb{C}$ be any two complex numbers, then the partial order relation $\precsim$ on $\mathbb{C}$ is defined as follows:

$z_1 \precsim z_2$ if and only if Re $(z_1) \leq$ Re (z_2) and Im $(z_1) \leq$ Im (z_2)

That is

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

$$\begin{aligned}
&(1) \quad \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2),\ \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\\
&(2) \quad \mathrm{Re}\,(z_1) < \mathrm{Re}\,(z_2),\ \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\\
&(3) \quad \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2),\ \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)\\
&(4) \quad \mathrm{Re}\,(z_1) < \mathrm{Re}\,(z_2),\ \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2)
\end{aligned}$$

In particular, we can say $z_1 \lnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (2), (3) and (4) is satisfied and $z_1 \prec z_2$ if only (4) is satisfied. We can easily check the following fundamental properties of partial order relation $\precsim$ on $\mathbb{C}$:

1. If $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$;
2. If $z_1 \precsim z_2, z_2 \prec z_3$then $z_1 \prec z_3$ and
3. If$z_1 \precsim z_2$ and $\lambda > 0$ is a real number then $\lambda z_1 \precsim \lambda z_2$

1.1 Complex valued metric space. Azam et al. (2011) define the complex valued metric space as

DEFINITION 1. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions*:

1. $0 \precsim d(x, y)$ *for all* $x, y \in X$
2. $d(x, y) = 0$ *if and only if* $x = y$
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$

Then d is called a complex valued metric on X and (X, d) is called the complex valued metric space.

DEFINITION 2. *Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions*:

1. $0 \precsim d(x, y)$ *for all* $x, y \in X$
2. $d(x, y) = 0$ *if and only if* $x = y$

3. $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *and*

4. $d(x,y) \precsim s\,[d(x,z) + d(z,y)]$ *for all* $x, y, z \in X$

Then d *is called a complex valued b-metric on* X *and* (X,d) *is called the complex valued b-metric space.*

1.2 Bicomplex Number. Segre (1892) defines the bicomplex number as:

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote $i_1 i_2 = j$, which is known as the hyperbolic unit and such that $j^2 = 1$, $i_1 j = j i_1 = -i_2$, $i_2 j = j i_2 = -i_1$. Also the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

or

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}\}.$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}$.

If $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is $\xi \pm \eta = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2 (z_2 \pm w_2)$ and the product is $\xi.\eta = (z_1 + i_2 z_2).(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2 (z_1 w_2 + z_2 w_1)$.

1.2.1 Idempotent representation of bicomplex number. There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$, and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be expressed as the combination of e_1 and e_2, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2)\, e_2.$$

This representation of ξ is known as the idempotent representation of bicomplex number and the complex coefficients $\xi_1 = (z_1 - i_1 z_2)$and $\xi_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number ξ.

1.2.2 Non-Singular and Singular elements. An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element η in $\mathbb{C}_2$ such that $\xi\eta = 1$ and η is said to be the inverse (multiplicative) of ξ. Consequently ξ is said to be the inverse (multiplicative) of η. An element which has an inverse in $\mathbb{C}_2$ is said to be the nonsingular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if and only if $|z_1^2 + z_2^2| \neq 0$ and singular if and only if $|z_1^2 + z_2^2| = 0$. and the inverse of ξ is defined as

$$\xi^{-1} = \eta = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}$$

Zero is the only one element in $\mathbb{R}$ which does not have any multiplicative inverse and in $\mathbb{C}$, $0 = 0 + i0$ is the only one element which does not have multiplicative inverse. We denote the set of singular elements of $\mathbb{R}$ and $\mathbb{C}$ by O_0 and O_1 respectively. But there are more than one element in $\mathbb{C}_2$ which does not have any multiplicative inverse; we denote this set by O_2 and clearly $O_0 = O_1 \subset O_2$.

1.2.3 Degenerated bicomplex number. A bicomplex number $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ is degenerated, in that case ξ^{-1} exists and it is also degenerated.

1.2.4 Norm of a bicomplex number. The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function, $\|\cdot\| : \mathbb{C}_2 \to \mathbb{R}^+$ is defined by

$$\begin{aligned} \|\xi\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$ then $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ holds instead of $\|\xi\eta\| \leq \|\xi\|\,\|\eta\|$, therefore $\mathbb{C}_2$ is not the Banach algebra.

Now we define the partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$ then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$

That is

$$\xi \precsim_{i_2} \eta$$

if one of the following conditions is satisfied:

$$(1) \quad z_1 = w_1,\ z_2 = w_2$$

$$(2) \quad z_1 \prec w_1,\ z_2 = w_2$$

$$(3) \quad z_1 = w_1,\ z_2 \prec w_2$$

$$(4) \quad z_1 \prec w_1,\ z_2 \prec w_2$$

In particular we can write $\xi \lnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$ i.e. one of (2), (3) and (4) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (4) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \Rightarrow \|\xi\| \leq \|\eta\|$

(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$

(iii) $\|a\xi\| = a\,\|\xi\|$ where a is a non negative real number.

(iv) $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$, the equality holds only when at least one of ξ and η is degenerated.

(v) $\|\xi^{-1}\| = \|\xi\|^{-1}$ if ξ is a degenerated bicomplex number with $0 \prec \xi$.

(vi) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$, if η is a degenerated bicomplex number.

Now we can define the bicomplex valued metric space as follows:

1.3 Bicomplex valued metric space

DEFINITION 3. (Choi, Datta, Biswas and Islam, 2017) *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x,y)$ *for all* $x, y \in X$
2. $d(x,y) = 0$ *if and only if* $x = y$
3. $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *and*
4. $d(x,y) \precsim_{i_2} d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

Then d is called a bicomplex valued metric on X and (X,d) is called the bicomplex valued metric space.

DEFINITION 4. *Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x,y)$ *for all* $x, y \in X$
2. $d(x,y) = 0$ *if and only if* $x = y$
3. $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *and*
4. $d(x,y) \precsim_{i_2} s\,[d(x,z) + d(z,y)]$ *for all* $x, y, z \in X$.

Then d is called a bicomplex valued b-metric on X and (X,d) is called the bicomplex valued b-metric space.

DEFINITION 5. (i) *Let $A \subseteq X$ and $a \in A$ is said to be an interior point of A if there exists a $0 \prec_{i_2} r \in \mathbb{C}_2$ such that*

$$B(a,r) = \{x \in X : d(a,x) \prec_{i_2} r\} \subseteq A$$

And the subset $A \subseteq X$ is said to be an open set if each point of A is an interior point of A.

(ii) *A point $a \in X$ is said to be a limit point of A if for all $0 \prec_{i_2} r \in \mathbb{C}_2$ such that*

$$B(a,r) \cap \{A - \{a\}\} \neq \varnothing$$

And the subset $A \subseteq X$ is said to be a closed set if all the limit points of A belong to A.

(iii) *The family*

$$F = \{B(a,r) : a \in X, 0 \prec_{i_2} r \in \mathbb{C}_2\}$$

is a sub-basis for a Hausdorff topology τ on X.

DEFINITION 6. *For a bicomplex valued metric space (X,d)*

(i) *A sequence $\{x_n\}$ in X is said to be a convergent sequence and converges to a point x if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec_{i_2} r$, for all $n > n_0$. And we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

(ii) *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in (X,d) if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$, for all $m, n \in \mathbb{N}$ and $n > n_0$.*

(iii) *If every cauchy sequence in X is convergent in X then (X,d) is said to be a complete bicomplex valued metric space.*

LEMMA 7. *Let (X,d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be convergent to a point x if and only if* $\lim_{n\to\infty} \|d(x_n,x)\| = 0$.

LEMMA 8. *Let (X,d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if and only if* $\lim_{n\to\infty} \|d(x_n,x_{n+m})\| = 0$.

2. Main Theorems. In this section we prove some fixed point theorems on bicomplex valued b -metric space for a pair of self contracting mappings.

THEOREM 9. *Let (X,d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$and $1+d(x,y)$ degenerated for all $x,y \in X$. Let the mappings $S,T : X \to X$ satisfy the condition*

$$d(Sx,Ty) \precsim i_2 A d(x,y) + B\frac{d(x,Sx)\,d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)\,d(x,Ty)}{1+d(x,y)}$$

$$+D\frac{d(x,Sx)\,d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Sx)\,d(y,Ty)}{1+d(x,y)} \quad (1)$$

for all $x,y \in X$ where A,B,C,D and E are non negative real numbers such that $A+\sqrt{2}\,(B+C+2sD+2sE) < 1$. Then S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X and we define a sequence $\{x_n\}$ such that

$$x_{2k+1} = Sx_{2k}, \quad x_{2k+2} = Tx_{2k+1}, \text{ for } k = 0,1,2,\cdots$$

Therefore by using inequality (1) we obtain

$$\begin{aligned} d(x_{2k+1},x_{2k+2}) &= d(Sx_{2k},Tx_{2k+1}) \\ &\precsim {}_{i_2} A d(x_{2k},x_{2k+1}) + B\frac{d(x_{2k},Sx_{2k})\,d(x_{2k+1},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} \\ &\quad +C\frac{d(x_{2k+1},Sx_{2k})\,d(x_{2k},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} \\ &\quad +D\frac{d(x_{2k},Sx_{2k})\,d(x_{2k},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} \\ &\quad +E\frac{d(x_{2k+1},Sx_{2k})\,d(x_{2k+1},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} \end{aligned}$$

$$
\begin{aligned}
&\precsim_{i_2} Ad\left(x_{2k}, x_{2k+1}\right) + B\frac{d\left(x_{2k}, x_{2k+1}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)} \\
&\quad + C\frac{d\left(x_{2k+1}, x_{2k+1}\right) d\left(x_{2k}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)} \\
&\quad + D\frac{d\left(x_{2k}, x_{2k+1}\right) d\left(x_{2k}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)} \\
&\quad + E\frac{d\left(x_{2k+1}, x_{2k+1}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)} \\
&\precsim_{i_2} Ad\left(x_{2k}, x_{2k+1}\right) + B\frac{d\left(x_{2k}, x_{2k+1}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)} \\
&\quad + D\frac{d\left(x_{2k}, x_{2k+1}\right) d\left(x_{2k}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \\
&\quad \leq A\left\|d\left(x_{2k}, x_{2k+1}\right)\right\| + \sqrt{2}B\frac{\left\|d\left(x_{2k}, x_{2k+1}\right)\right\|}{\left\|1 + d\left(x_{2k}, x_{2k+1}\right)\right\|}\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \\
&\qquad + \sqrt{2}D\frac{\left\|d\left(x_{2k}, x_{2k+1}\right)\right\|}{\left\|1 + d\left(x_{2k}, x_{2k+1}\right)\right\|}\left\|d\left(x_{2k}, x_{2k+2}\right)\right\|
\end{aligned}
$$

Since, $\left\|d\left(x_{2k}, x_{2k+1}\right)\right\| \leq \left\|1 + d\left(x_{2k}, x_{2k+1}\right)\right\|$

Therefore,

$$
\begin{aligned}
&\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \\
&\quad \leq A\left\|d\left(x_{2k}, x_{2k+1}\right)\right\| + \sqrt{2}B\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| + \sqrt{2}D\left\|d\left(x_{2k}, x_{2k+2}\right)\right\| \\
&\quad \leq A\left\|d\left(x_{2k}, x_{2k+1}\right)\right\| + \sqrt{2}B\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| + \sqrt{2}sD\left\|d\left(x_{2k}, x_{2k+1}\right)\right\| \\
&\quad + \sqrt{2}sD\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \\
\Rightarrow &\quad \left(1 - \sqrt{2}\left(B + sD\right)\right)\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \leq \left(A + \sqrt{2}sD\right)\left\|d\left(x_{2k}, x_{2k+1}\right)\right\| \\
\Rightarrow &\quad \left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \leq \frac{A + \sqrt{2}sD}{1 - \sqrt{2}\left(B + sD\right)}\left\|d\left(x_{2k}, x_{2k+1}\right)\right\|
\end{aligned}
$$

Similarly,

$$\begin{aligned}
d\left(x_{2k+2}, x_{2k+3}\right) &= d\left(Tx_{2k+1}, Sx_{2k+2}\right) = d\left(Sx_{2k+2}, Tx_{2k+1}\right) \\
&\precsim_{i_2} Ad\left(x_{2k+2}, x_{2k+1}\right) + B\frac{d\left(x_{2k+2}, Sx_{2k+2}\right) d\left(x_{2k+1}, Tx_{2k+1}\right)}{1 + d\left(x_{2k+2}, x_{2k+1}\right)} \\
&\quad + C\frac{d\left(x_{2k+1}, Sx_{2k+2}\right) d\left(x_{2k+2}, Tx_{2k+1}\right)}{1 + d\left(x_{2k+2}, x_{2k+1}\right)} \\
&\quad + D\frac{d\left(x_{2k+2}, Sx_{2k+2}\right) d\left(x_{2k+2}, Tx_{2k+1}\right)}{1 + d\left(x_{2k+2}, x_{2k+1}\right)} \\
&\quad + E\frac{d\left(x_{2k+1}, Sx_{2k+2}\right) d\left(x_{2k+1}, Tx_{2k+1}\right)}{1 + d\left(x_{2k+2}, x_{2k+1}\right)} \\
&\precsim_{i_2} Ad\left(x_{2k+2}, x_{2k+1}\right) + B\frac{d\left(x_{2k+2}, x_{2k+3}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k+1}, x_{2k+2}\right)} \\
&\quad + C\frac{d\left(x_{2k+1}, x_{2k+3}\right) d\left(x_{2k+2}, x_{2k+2}\right)}{1 + d\left(x_{2k+1}, x_{2k+2}\right)} \\
&\quad + D\frac{d\left(x_{2k+2}, x_{2k+3}\right) d\left(x_{2k+2}, x_{2k+2}\right)}{1 + d\left(x_{2k+1}, x_{2k+2}\right)} \\
&\quad + E\frac{d\left(x_{2k+1}, x_{2k+3}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k+1}, x_{2k+2}\right)} \\
&\precsim_{i_2} Ad\left(x_{2k+2}, x_{2k+1}\right) + B\frac{d\left(x_{2k+2}, x_{2k+3}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k+1}, x_{2k+2}\right)} \\
&\quad E\frac{d\left(x_{2k+1}, x_{2k+3}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k+1}, x_{2k+2}\right)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
&\quad \leq A\left\|d\left(x_{2k+2}, x_{2k+1}\right)\right\| + \sqrt{2}B\frac{\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|}{\left\|1 + d\left(x_{2k+1}, x_{2k+2}\right)\right\|}\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
&\qquad + \sqrt{2}E\frac{\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|}{\left\|1 + d\left(x_{2k+1}, x_{2k+2}\right)\right\|}\left\|d\left(x_{2k+1}, x_{2k+3}\right)\right\|
\end{aligned}$$

Since, $\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \leq \left\|1 + d\left(x_{2k+1}, x_{2k+2}\right)\right\|$

Therefore,

$$\begin{aligned}
&\|d(x_{2k+2},x_{2k+3})\|\\
&\quad\le A\|d(x_{2k+2},x_{2k+1})\|+\sqrt{2}B\|d(x_{2k+2},x_{2k+3})\|+\sqrt{2}E\|d(x_{2k+1},x_{2k+3})\|\\
&\quad\le A\|d(x_{2k+2},x_{2k+1})\|+\sqrt{2}B\|d(x_{2k+2},x_{2k+3})\|+\sqrt{2}sE\|d(x_{2k+1},x_{2k+2})\|\\
&\qquad+\sqrt{2}sE\|d(x_{2k+2},x_{2k+3})\|\\
\Rightarrow\;&\left(1-\sqrt{2}(B+sE)\right)\|d(x_{2k+2},x_{2k+3})\|\le\left(A+\sqrt{2}sE\right)\|d(x_{2k+1},x_{2k+2})\|\\
\Rightarrow\;&\|d(x_{2k+2},x_{2k+3})\|\le\frac{A+\sqrt{2}sE}{1-\sqrt{2}(B+sE)}\|d(x_{2k+1},x_{2k+2})\|
\end{aligned}$$

Since $A+\sqrt{2}(B+C+2sD+2sE)<1$, therefore $\frac{A+\sqrt{2}sD}{1-\sqrt{2}(B+sD)}<1$ and $\frac{A+\sqrt{2}sE}{1-\sqrt{2}(B+sE)}<1$. Let $\alpha=\max\left\{\frac{A+\sqrt{2}sD}{1-\sqrt{2}(B+sD)},\frac{A+\sqrt{2}sE}{1-\sqrt{2}(B+sE)}\right\}$, then $\alpha<1$ and

$$\begin{aligned}
&\|d(x_{n+1},x_{n+2})\|\\
&\quad\le\alpha\|d(x_n,x_{n+1})\| \qquad (2)\\
&\quad\le\alpha^2\|d(x_{n-1},x_n)\|\cdots\le\alpha^{n+1}\|d(x_0,x_1)\|\quad\text{for all}\quad n=0,1,2, \qquad (3)
\end{aligned}$$

Then for any two positive integers m,n with $m>n$

$$d(x_n,x_m)\precsim_{i_2}s\left[d(x_n,x_{n+1})+d(x_{n+1},x_m)\right].$$

Therefore,

$$\begin{aligned}
&\|d(x_n,x_m)\|\\
&\quad\le s\|d(x_n,x_{n+1})\|+s\|d(x_{n+1},x_m)\|\\
&\quad\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^2\|d(x_{n+2},x_m)\|\\
&\quad\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|\\
&\qquad\quad+s^3\|d(x_{n+2},x_{n+3})\|+s^3\|d(x_{n+3},x_m)\|.\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Rightarrow\;&\|d(x_n,x_m)\|
\end{aligned}$$

$$\leq s\left\|d\left(x_n,x_{n+1}\right)\right\|+s^2\left\|d\left(x_{n+1},x_{n+2}\right)\right\|+s^3\left\|d\left(x_{n+2},x_{n+3}\right)\right\|+\cdots$$
$$\cdots+s^{m-n-1}\left\|d\left(x_{m-1},x_m\right)\right\|$$
$$\Rightarrow \left\|d\left(x_n,x_m\right)\right\|$$
$$\leq s\left\|d\left(x_n,x_{n+1}\right)\right\|+s^2\left\|d\left(x_{n+1},x_{n+2}\right)\right\|+s^3\left\|d\left(x_{n+2},x_{n+3}\right)\right\|+\cdots$$
$$\cdots+s^{m-n}\left\|d\left(x_{m-1},x_m\right)\right\|$$

Therefore by using (3)

$$\left\|d\left(x_n,x_m\right)\right\| \leq s\alpha^n\left\|d\left(x_0,x_1\right)\right\|+s^2\alpha^{n+1}\left\|d\left(x_0,x_1\right)\right\|+s^3\alpha^{n+2}\left\|d\left(x_0,x_1\right)\right\|$$
$$+\cdots+s^{m-n}\alpha^{m-1}\left\|d\left(x_0,x_1\right)\right\|.$$

$$\Rightarrow \left\|d\left(x_n,x_m\right)\right\| \leq \sum_{i=1}^{m-n} s^i\alpha^{i+n-1}\left\|d\left(x_0,x_1\right)\right\|.$$

$$\Rightarrow \left\|d\left(x_n,x_m\right)\right\| \leq \sum_{i=1}^{m-n} s^{i+n-1}\alpha^{i+n-1}\left\|d\left(x_0,x_1\right)\right\|, \text{ since } s \geq 1.$$

$$\Rightarrow \left\|d\left(x_n,x_m\right)\right\| \leq \sum_{j=n}^{m-1} s^j\alpha^j\left\|d\left(x_0,x_1\right)\right\|$$

$$\Rightarrow \left\|d\left(x_n,x_m\right)\right\| \leq \sum_{j=n}^{\infty} (s\alpha)^j\left\|d\left(x_0,x_1\right)\right\|$$

$$\Rightarrow \left\|d\left(x_n,x_m\right)\right\| \leq \frac{(s\alpha)^n}{1-s\alpha}\left\|d\left(x_0,x_1\right)\right\|.$$

Since $\frac{(s\alpha)^n}{1-s\alpha} \longrightarrow 0$ as $n \longrightarrow \infty$. Hence for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\left\|d\left(x_n,x_m\right)\right\| < \varepsilon,\quad$ for all $m,n > n_0$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete bicomplex valued b-metric space, then there exists $u \in X$ such that $\lim_{n\longrightarrow\infty} x_n = u$.

Now we show that $u = Su$, if not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d\left(u,Su\right) = \xi$.

Therefore,

$$\xi = d\left(u,Su\right)$$
$$\precsim_{i_2} sd\left(u,x_{2k+2}\right)+sd\left(x_{2k+2},Su\right)$$

$$\begin{aligned}
&\precsim_{i_2} sd(u, x_{2k+2}) + sd(Su, Tx_{2k+1}) \\
&\precsim_{i_2} sd(u, x_{2k+2}) + sAd(u, x_{2k+1}) + sB\frac{d(u, Su)\, d(x_{2k+1}, Tx_{2k+1})}{1 + d(u, x_{2k+1})} \\
&\quad + sC\frac{d(x_{2k+1}, Su)\, d(u, Tx_{2k+1})}{1 + d(u, x_{2k+1})} + sD\frac{d(u, Su)\, d(u, Tx_{2k+1})}{1 + d(u, x_{2k+1})} \\
&\quad + sE\frac{d(x_{2k+1}, Su)\, d(x_{2k+1}, Tx_{2k+1})}{1 + d(u, x_{2k+1})}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \xi &\precsim i_2 sd(u, x_{2k+2}) + sAd(u, x_{2k+1}) + sB\frac{d(u, Su)\, d(x_{2k+1}, x_{2k+2})}{1 + d(u, x_{2k+1})} \\
&\quad + sC\frac{d(x_{2k+1}, Su)\, d(u, x_{2k+2})}{1 + d(u, x_{2k+1})} + sD\frac{d(u, Su)\, d(u, x_{2k+2})}{1 + d(u, x_{2k+1})} \\
&\quad + sE\frac{d(x_{2k+1}, Su)\, d(x_{2k+1}, x_{2k+2})}{1 + d(u, x_{2k+1})}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\xi\| &\leq s\,\|d(u, x_{2k+2})\| + sA\,\|d(u, x_{2k+1})\| + \sqrt{2}sB\frac{\|(u, Su)\|\,\|d(x_{2k+1}, x_{2k+2})\|}{\|1 + d(u, x_{2k+1})\|} \\
&\quad + \sqrt{2}sC\frac{\|d(x_{2k+1}, Su)\|\,\|d(u, x_{2k+2})\|}{\|1 + d(u, x_{2k+1})\|} + \sqrt{2}sD\frac{\|d(u, Su)\|\,\|d(u, x_{2k+2})\|}{\|1 + d(u, x_{2k+1})\|} \\
&\quad + \sqrt{2}sE\frac{\|d(x_{2k+1}, Su)\|\,\|d(x_{2k+1}, x_{2k+2})\|}{\|1 + d(u, x_{2k+1})\|}
\end{aligned}$$

Since $\lim_{n \longrightarrow \infty} x_n = u$, taking limit on both sides as $n \longrightarrow \infty$ we get, $\|\xi\| \leq 0$, which is a contradiction, hence $\|\xi\| = 0 \Rightarrow \|d(u, Su)\| = 0 \Rightarrow u = Su$. Similarly, we can show that $u = Tu$.

Therefore S and T have a common fixed point.

Now, we show that S and T have a unique common fixed point, for this let u^* be another common fixed point of S and T in X.

Then

$$\begin{aligned}
d(u, u^*) &= d(Su, Tu^*) \\
&\precsim_{i_2} Ad(u, u^*) + B\frac{d(u, Su)\, d(u^*, Tu^*)}{1 + d(u, u^*)}
\end{aligned}$$

$$+C\frac{d\left(u^{*},Su\right)d\left(u,Tu^{*}\right)}{1+d\left(u,u^{*}\right)}+D\frac{d\left(u,Su\right)d\left(u,Tu^{*}\right)}{1+d\left(u,u^{*}\right)}$$
$$+E\frac{d\left(u^{*},Su\right)d\left(u^{*},Tu^{*}\right)}{1+d\left(u,u^{*}\right)}$$

Therefore,

$$\begin{aligned}\left\|d\left(u,u^{*}\right)\right\| \leq\ & A\left\|d\left(u,u^{*}\right)\right\|+\sqrt{2}B\frac{\left\|d\left(u,Su\right)\right\|\left\|d\left(u^{*},Tu^{*}\right)\right\|}{\left\|1+d\left(u,u^{*}\right)\right\|}\\ &+\sqrt{2}C\frac{\left\|d\left(u^{*},Su\right)\right\|\left\|d\left(u,Tu^{*}\right)\right\|}{\left\|1+d\left(u,u^{*}\right)\right\|}+\sqrt{2}D\frac{\left\|d\left(u,Su\right)\right\|\left\|d\left(u,Tu^{*}\right)\right\|}{\left\|1+d\left(u,u^{*}\right)\right\|}\\ &+\sqrt{2}E\frac{\left\|d\left(u^{*},Su\right)\right\|\left\|d\left(u^{*},Tu^{*}\right)\right\|}{\left\|1+d\left(u,u^{*}\right)\right\|}\\ \leq\ & A\left\|d\left(u,u^{*}\right)\right\|+\sqrt{2}C\frac{\left\|d\left(u^{*},Su\right)\right\|\left\|d\left(u,Tu^{*}\right)\right\|}{\left\|1+d\left(u,u^{*}\right)\right\|}\\ \leq\ & A\left\|d\left(u,u^{*}\right)\right\|+\sqrt{2}C\frac{\left\|d\left(u^{*},u\right)\right\|}{\left\|1+d\left(u,u^{*}\right)\right\|}\left\|d\left(u,u^{*}\right)\right\|\\ &\Rightarrow \left\|d\left(u,u^{*}\right)\right\|\leq\left(A+\sqrt{2}C\right)\left\|d\left(u,u^{*}\right)\right\|\\ &\Rightarrow \left\|d\left(u,u^{*}\right)\right\|=0\\ &\Rightarrow u=u^{*}\end{aligned}$$

This completes the proof of the theorem.

COROLLARY 10. *Let (X,d) be a complete bicomplex valued b-metric space with the coefficient $s\geq 1$and $1+d(x,y)$ degenerated for all $x,y\in X$. Let the mapping $S:X\rightarrow X$ satisfies the condition*

$$d\left(Sx,Ty\right)\precsim_{i_2}Ad\left(x,y\right)+B\frac{d\left(x,Sx\right)d\left(y,Sy\right)}{1+d\left(x,y\right)}+C\frac{d\left(y,Sx\right)d\left(x,Sy\right)}{1+d\left(x,y\right)}$$
$$+D\frac{d\left(x,Sx\right)d\left(x,Sy\right)}{1+d\left(x,y\right)}+E\frac{d\left(y,Sx\right)d\left(y,Sy\right)}{1+d\left(x,y\right)}$$

for all $x,y\in X$ where A,B,C,D and E are non negative real numbers such that $A+\sqrt{2}\left(B+C+2sD+2sE\right)<1$. Then S has a unique fixed point in X.

Proof: We can easily prove this result by applying the Theorem 9 and taking $T = S$.

By choosing $E = 0$ in Theorem 9 we get the following corollary.

COROLLARY 11. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$and $1 + d(x, y)$ degenerated for all $x, y \in X$. Let the mappings $S, T : X \to X$ satisfy the condition*

$$d(Sx, Ty) \precsim_{i_2} Ad(x, y) + B\frac{d(x, Sx)\, d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)\, d(x, Ty)}{1 + d(x, y)} + D\frac{d(x, Sx)\, d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$ where A, B, C and D are non negative real numbers such that $A + \sqrt{2}(B + C + 2sD) < 1$. Then S and T have a unique common fixed point in X.

COROLLARY 12. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$and $1+d(x, y)$ degenerated for all $x, y \in X$. Let the mapping $S : X \to X$ satisfies the condition*

$$d(Sx, Ty) \precsim_{i_2} Ad(x, y) + B\frac{d(x, Sx)\, d(y, Sy)}{1 + d(x, y)} + C\frac{d(y, Sx)\, d(x, Sy)}{1 + d(x, y)} + D\frac{d(x, Sx)\, d(x, Sy)}{1 + d(x, y)}$$

for all $x, y \in X$ where A, B, C and D are non negative real numbers such that $A + \sqrt{2}(B + C + 2sD) < 1$. Then S has a unique fixed point in X.

Proof: This can be proved by putting $T = S$ in Corollary 11.

By choosing $D = 0$ in Theorem 9 we get the following corollary.

COROLLARY 13. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$and $1 + d(x, y)$ degenerated for all $x, y \in X$. Let the mappings $S, T : X \to X$ satisfy the condition*

$$d(Sx, Ty) \precsim_{i_2} Ad(x, y) + B\frac{d(x, Sx)\, d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)\, d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Sx)\, d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$ where A, B, C and E are non negative real numbers such that $A + \sqrt{2}(B + C + 2sE) < 1$. Then S and T have a unique common fixed point in X.

COROLLARY 14. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$ and $1+d(x, y)$ degenerated for all $x, y \in X$. Let the mapping $S : X \to X$ satisfies the condition*

$$d(Sx, Ty) \precsim_{i_2} Ad(x, y) + B\frac{d(x, Sx)\, d(y, Sy)}{1 + d(x, y)} + C\frac{d(y, Sx)\, d(x, Sy)}{1 + d(x, y)} + E\frac{d(y, Sx)\, d(y, Sy)}{1 + d(x, y)}$$

for all $x, y \in X$ where A, B, C and E are non negative real numbers such that $A + \sqrt{2}(B + C + 2sE) < 1$. Then S has a unique fixed point in X.

Proof: This can be proved by putting $T = S$ in Corollary 13.

COROLLARY 15. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$ and $1+d(x, y)$ degenerated for all $x, y \in X$. Let the mapping $S : X \to X$ satisfies the condition*

$$d(S^n x, S^n y) \precsim_{i_2} Ad(x, y) + B\frac{d(x, S^n x)\, d(y, S^n y)}{1 + d(x, y)} + C\frac{d(y, S^n x)\, d(x, S^n y)}{1 + d(x, y)} + D\frac{d(x, S^n x)\, d(x, S^n y)}{1 + d(x, y)} + E\frac{d(y, S^n x)\, d(y, S^n y)}{1 + d(x, y)}$$

for all $x, y \in X$ where A, B, C, D and E are non negative real numbers such that $A + \sqrt{2}(B + C + 2sD + 2sE) < 1$. Then S has a unique fixed point in X.

Proof: By Corollary 10 there exists a unique point $u \in X$ such that

$$S^n u = u.$$

Therefore,

$$\begin{aligned} d(Su, u) &= d(SS^n u, S^n u) = d(S^n Su, S^n u) \\ &\precsim i_2 Ad(Su, u) + B\frac{d(Su, S^n Su)\, d(u, S^n u)}{1 + d(Su, u)} + C\frac{d(u, S^n Su)\, d(Su, S^n u)}{1 + d(Su, u)} \\ &\quad + D\frac{d(Su, S^n Su)\, d(Su, S^n u)}{1 + d(Su, u)} + E\frac{d(u, S^n Su)\, d(u, S^n u)}{1 + d(Su, u)} \\ &\precsim i_2 Ad(Su, u) + B\frac{d(Su, Su)\, d(u, u)}{1 + d(Su, u)} + C\frac{d(u, Su)\, d(Su, u)}{1 + d(Su, u)} \end{aligned}$$

$$+D\frac{d(Su,Su)\,d(Su,u)}{1+d(Su,u)}+E\frac{d(u,Su)\,d(u,u)}{1+d(Su,u)}$$

$$\precsim i_2 Ad(Su,u)+C\frac{d(u,Su)\,d(Su,u)}{1+d(Su,u)}$$

$$\Rightarrow \ \|d(Su,u)\| \le A\,\|d(Su,u)\|+\sqrt{2}C\,\|d(Su,u)\|\,\frac{\|d(u,Su)\|}{\|1+d(Su,u)\|}$$

$$\Rightarrow \ \|d(Su,u)\| \le \left(A+\sqrt{2}C\right)\|d(Su,u)\|$$

$$\Rightarrow \ \|d(Su,u)\| = 0$$

$$\Rightarrow \ Su=u.$$

This completes the proof of the corollary.

THEOREM 16. *Let (X,d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$and $1+d(x,y)$ degenerated for all $x,y \in X$. Let the mappings $S,T : X \to X$ satisfy the condition*

$$d(Sx,Ty) \precsim {}_{i_2} Ad(x,y)+B\frac{d(x,Sx)\,d(y,Ty)}{1+d(x,y)}$$
$$+C\left[d(x,Sx)+d(y,Ty)\right]+D\left[d(x,Ty)+d(y,Sx)\right] \tag{4}$$

for all $x,y \in X$ where A,B,C and D are non negative real numbers such that $A+\sqrt{2}B+2C+2sD<1$. Then S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X and we define a sequence $\{x_n\}$ such that

$$x_{2k+1}=Sx_{2k}, \quad x_{2k+2}=Tx_{2k+1}, \ \text{for} \ k=0,1,2,\cdots$$

Therefore by using inequality (4) we obtain

$$\begin{aligned} d(x_{2k+1},x_{2k+2}) &= d(Sx_{2k},Tx_{2k+1}) \\ &\precsim {}_{i_2} Ad(x_{2k},x_{2k+1})+B\frac{d(x_{2k},Sx_{2k})\,d(x_{2k+1},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} \\ &\quad +C\left[d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1})\right] \\ &\quad +D\left[d(x_{2k},Tx_{2k+1})+d(x_{2k+1},Sx_{2k})\right] \\ &\precsim {}_{i_2} Ad(x_{2k},x_{2k+1})+B\frac{d(x_{2k},x_{2k+1})\,d(x_{2k+1},x_{2k+2})}{1+d(x_{2k},x_{2k+1})} \end{aligned}$$

$$
\begin{aligned}
&+C\left[d\left(x_{2k},x_{2k+1}\right)+d\left(x_{2k+1},x_{2k+2}\right)\right]\\
&+Dd\left(x_{2k},x_{2k+2}\right)\\
\precsim_{i_2}\, &Ad\left(x_{2k},x_{2k+1}\right)+B\frac{d\left(x_{2k},x_{2k+1}\right)d\left(x_{2k+1},x_{2k+2}\right)}{1+d\left(x_{2k},x_{2k+1}\right)}\\
&+C\left[d\left(x_{2k},x_{2k+1}\right)+d\left(x_{2k+1},x_{2k+2}\right)\right]\\
&+sD\left[d\left(x_{2k},x_{2k+1}\right)+d\left(x_{2k+1},x_{2k+2}\right)\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\\
&\quad\leq A\left\|d\left(x_{2k},x_{2k+1}\right)\right\|+\sqrt{2}B\frac{\left\|d\left(x_{2k},x_{2k+1}\right)\right\|}{\left\|1+d\left(x_{2k},x_{2k+1}\right)\right\|}\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\\
&\qquad+C\left[\left\|d\left(x_{2k},x_{2k+1}\right)\right\|+\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\right]\\
&\qquad+sD\left[\left\|d\left(x_{2k},x_{2k+1}\right)\right\|+\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\right]
\end{aligned}
$$

Since, $\left\|d\left(x_{2k},x_{2k+1}\right)\right\|\leq\left\|1+d\left(x_{2k},x_{2k+1}\right)\right\|$
Therefore,

$$
\begin{aligned}
&\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\\
&\quad\leq A\left\|d\left(x_{2k},x_{2k+1}\right)\right\|+\sqrt{2}B\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\\
&\qquad+C\left[\left\|d\left(x_{2k},x_{2k+1}\right)\right\|+\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\right]\\
&\qquad+sD\left[\left\|d\left(x_{2k},x_{2k+1}\right)\right\|+\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\right]\\
\Rightarrow\ &\left(1-\sqrt{2}B-C-sD\right)\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\leq\left(A+C+sD\right)\left\|d\left(x_{2k},x_{2k+1}\right)\right\|\\
\Rightarrow\ &\left\|d\left(x_{2k+1},x_{2k+2}\right)\right\|\leq\frac{\left(A+C+sD\right)}{\left(1-\sqrt{2}B-C-sD\right)}\left\|d\left(x_{2k},x_{2k+1}\right)\right\|
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
d\left(x_{2k+2},x_{2k+3}\right)&=d\left(Tx_{2k+1,},Sx_{2k+2}\right)=d\left(Sx_{2k+2},Tx_{2k+1,}\right)\\
&\precsim_{i_2}Ad\left(x_{2k+2},x_{2k+1}\right)+B\frac{d\left(x_{2k+2},Sx_{2k+2}\right)d\left(x_{2k+1},Tx_{2k+1}\right)}{1+d\left(x_{2k+2},x_{2k+1}\right)}
\end{aligned}
$$

$$\begin{aligned}
&+C\left[d\left(x_{2k+2}, Sx_{2k+2}\right)+d\left(x_{2k+1}, Tx_{2k+1}\right)\right] \\
&+D\left[d\left(x_{2k+2}, Tx_{2k+1}\right)+d\left(x_{2k+1}, Sx_{2k+2}\right)\right] \\
\precsim_{i_2} & Ad\left(x_{2k+2}, x_{2k+1}\right)+B\frac{d\left(x_{2k+2}, x_{2k+3}\right)d\left(x_{2k+1}, x_{2k+2}\right)}{1+d\left(x_{2k+2}, x_{2k+1}\right)} \\
&+C\left[d\left(x_{2k+2}, x_{2k+3}\right)+d\left(x_{2k+1}, x_{2k+2}\right)\right] \\
&+D\left[d\left(x_{2k+2}, x_{2k+2}\right)+d\left(x_{2k+1}, x_{2k+3}\right)\right] \\
\precsim_{i_2} & Ad\left(x_{2k+2}, x_{2k+1}\right)+B\frac{d\left(x_{2k+2}, x_{2k+3}\right)d\left(x_{2k+1}, x_{2k+2}\right)}{1+d\left(x_{2k+2}, x_{2k+1}\right)} \\
&+C\left[d\left(x_{2k+2}, x_{2k+3}\right)+d\left(x_{2k+1}, x_{2k+2}\right)\right] \\
&+sD\left[+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(x_{2k+2}, x_{2k+3}\right)\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
&\quad\leq A\left\|d\left(x_{2k+2}, x_{2k+1}\right)\right\|+\sqrt{2}B\frac{\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|}{\left\|1+d\left(x_{2k+1}, x_{2k+2}\right)\right\|}\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
&\qquad+C\left[\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\|+\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|\right] \\
&\qquad+sD\left[\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|+\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\|\right]
\end{aligned}$$

Since, $\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \leq \left\|1+d\left(x_{2k+1}, x_{2k+2}\right)\right\|$

Therefore,

$$\begin{aligned}
&\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
&\quad\leq A\left\|d\left(x_{2k+2}, x_{2k+1}\right)\right\|+\sqrt{2}B\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
&\qquad C\left[\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\|+\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|\right] \\
&\qquad+sD\left[\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|+\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\|\right] \\
\Rightarrow &\ \left(1-\sqrt{2}B-C-sD\right)\left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \leq (A+C+sD)\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\| \\
\Rightarrow &\ \left\|d\left(x_{2k+2}, x_{2k+3}\right)\right\| \leq \frac{(A+C+sD)}{\left(1-\sqrt{2}B-C-sD\right)}\left\|d\left(x_{2k+1}, x_{2k+2}\right)\right\|
\end{aligned}$$

Since $A+\sqrt{2}B+2C+2sD<1$, therefore $\frac{(A+C+sD)}{(1-\sqrt{2}B-C-sD)}<1$. Let $\alpha=\frac{(A+C+sD)}{(1-\sqrt{2}B-C-sD)}$, then $\alpha<1$ and

$$\begin{aligned}\|d(x_{n+1},x_{n+2})\| &\leq \alpha\|d(x_n,x_{n+1})\| \\ &\leq \alpha^2\|d(x_{n-1},x_n)\|\cdots\leq\alpha^{n+1}\|d(x_0,x_1)\|. \text{ for all } n=0,1,2,\cdots\end{aligned}$$

Then for any two positive integers m,n with $m>n$

$$d(x_n,x_m)\precsim_{i_2} s[d(x_n,x_{n+1})+d(x_{n+1},x_m)].$$

Therefore,

$$\begin{aligned}&\|d(x_n,x_m)\| \\ &\leq s\|d(x_n,x_{n+1})\|+s\|d(x_{n+1},x_m)\| \\ &\leq s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^2\|d(x_{n+2},x_m)\| \\ &\leq s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\| \\ &\quad +s^3\|d(x_{n+2},x_{n+3})\|+s^3\|d(x_{n+3},x_m)\|. \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\|d(x_n,x_m)\| \\ &\leq s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^3\|d(x_{n+2},x_{n+3})\|+\cdots \\ &\quad +s^{m-n-1}\|d(x_{m-1},x_m)\| \\ \Rightarrow\ &\|d(x_n,x_m)\| \\ &\leq s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^3\|d(x_{n+2},x_{n+3})\|+\cdots \\ &\quad \cdots+s^{m-n}\|d(x_{m-1},x_m)\|\end{aligned}$$

Therefore by using (3)

$$\begin{aligned}\|d(x_n,x_m)\| \leq\ & s\alpha^n\|d(x_0,x_1)\|+s^2\alpha^{n+1}\|d(x_0,x_1)\|+s^3\alpha^{n+2}\|d(x_0,x_1)\|+\cdots \\ &\cdots+s^{m-n}\alpha^{m-1}\|d(x_0,x_1)\|.\end{aligned}$$

$$\Rightarrow \quad \|d(x_n, x_m)\| \leq \sum_{i=1}^{m-n} s^i \alpha^{i+n-1} \|d(x_0, x_1)\|.$$

$$\Rightarrow \quad \|d(x_n, x_m)\| \leq \sum_{i=1}^{m-n} s^{i+n-1} \alpha^{i+n-1} \|d(x_0, x_1)\|, \quad \text{since } \ s \geq 1.$$

$$\Rightarrow \quad \|d(x_n, x_m)\| \leq \sum_{j=n}^{m-1} s^j \alpha^j \|d(x_0, x_1)\|$$

$$\Rightarrow \quad \|d(x_n, x_m)\| \leq \sum_{j=n}^{\infty} (s\alpha)^j \|d(x_0, x_1)\|$$

$$\Rightarrow \quad \|d(x_n, x_m)\| \leq \frac{(s\alpha)^n}{1 - s\alpha} \|d(x_0, x_1)\|.$$

Since $\frac{(s\alpha)^n}{1-s\alpha} \longrightarrow 0$ as $n \longrightarrow \infty$. Hence for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|d(x_n, x_m)\| < \varepsilon, \quad$ for all $m, n > n_0$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete bicomplex valued b-metric space, then there exists $u \in X$ such that $\lim_{n \longrightarrow \infty} x_n = u$.

Now we show that $u = Su$, if not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u, Su) = \xi$.

Therefore,

$$\begin{aligned}
\xi &= d(u, Su) \\
&\precsim i_2 s d(u, x_{2k+2}) + s d(x_{2k+2}, Su) \\
&\precsim i_2 s d(u, x_{2k+2}) + s d(Su, T x_{2k+1}) \\
&\precsim i_2 s d(u, x_{2k+2}) + s A d(u, x_{2k+1}) + s B \frac{d(u, Su)\, d(x_{2k+1}, T x_{2k+1})}{1 + d(u, x_{2k+1})} \\
&\qquad + s C \left[d(u, Su) + d(x_{2k+1}, T x_{2k+1})\right] + s D \left[d(u, T x_{2k+1}) + d(x_{2k+1}, Su)\right] \\
&\precsim i_2 s d(u, x_{2k+2}) + s A d(u, x_{2k+1}) + s B \frac{d(u, Su)\, d(x_{2k+1}, x_{2k+2})}{1 + d(u, x_{2k+1})} \\
&\qquad + s C \left[d(u, Su) + d(x_{2k+1}, x_{2k+2})\right] \\
&\qquad + s D \left[d(u, x_{2k+2}) + d(x_{2k+1}, Su)\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\xi\| \leq\ & s\,\|d\,(u,x_{2k+2})\| + sA\,\|d\,(u,x_{2k+1})\| \\ & +\sqrt{2}sB\frac{\|(u,Su)\|\,\|d\,(x_{2k+1},x_{2k+2})\|}{\|1+d\,(u,x_{2k+1})\|} \\ & +sC\,[\|d\,(u,Su)\| + \|d\,(x_{2k+1},x_{2k+2})\|] \\ & +sD\,[\|d\,(u,x_{2k+2})\| + \|d\,(x_{2k+1},Su)\|]\end{aligned}$$

Since $\lim_{n\longrightarrow\infty} x_n = u$, taking limit on both sides as $n \longrightarrow \infty$ we get,

$$\|\xi\| \leq s\,(C+D)\,\|d\,(u,Su)\|\,.$$

Which is a contradiction, hence $\|\xi\| = 0 \Rightarrow \|d\,(u,Su)\| = 0 \Rightarrow u = Su$. Similarly, we can show that $u = Tu$.

Therefore S and T have a common fixed point.

Now, we show that S and T have a unique common fixed point, for this let u^* be another common fixed point of S and T in X.

Then

$$\begin{aligned}d\,(u,u^*) =\ & d\,(Su,Tu^*) \\ & \precsim i_2Ad\,(u,u^*) + B\frac{d\,(u,Su)\,d\,(u^*,Tu^*)}{1+d\,(u,u^*)} \\ & \quad +C\,[d\,(u,Su)+d\,(u^*,Tu^*)] + D\,[d\,(u,Tu^*)+d\,(u^*,Su)]\end{aligned}$$

Therefore,

$$\begin{aligned}\|d\,(u,u^*)\| \leq\ & A\,\|d\,(u,u^*)\| + \sqrt{2}B\frac{\|d\,(u,Su)\|\,\|d\,(u^*,Tu^*)\|}{\|1+d\,(u,u^*)\|} \\ & +C\,[\|d\,(u,Su)\| + \|d\,(u^*,Tu^*)\|] \\ & +D\,[\|d\,(u,Tu^*)\| + \|d\,(u^*,Su)\|] \\ \leq\ & A\,\|d\,(u,u^*)\| + D\,[\|d\,(u,u^*)\| + \|d\,(u^*,u)\|] \\ \Rightarrow\ & \|d\,(u,u^*)\| \leq (A+2D)\,\|d\,(u,u^*)\| \\ \Rightarrow\ & \|d\,(u,u^*)\| = 0 \\ \Rightarrow\ & u = u^*\end{aligned}$$

This proves the theorem.

COROLLARY 17. *Let (X,d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$ and $1+d(x,y)$ degenerated for all $x,y \in X$. Let the mapping $S : X \to X$ satisfies the condition*

$$d(Sx,Sy) \precsim i_2 Ad(x,y) + B\frac{d(x,Sx)\,d(y,Sy)}{1+d(x,y)} \\ +C[d(x,Sx)+d(y,Sy)] + D[d(x,Sy)+d(y,Sx)]$$

for all $x,y \in X$ where A,B,C and D are non negative real numbers such that $A+\sqrt{2}B+2C+2sD<1$. Then S and T have a unique common fixed point in X.

Proof: We can easily prove this result by applying the Theorem 16 and taking $T=S$.

COROLLARY 18. *Let (X,d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$ and $1+d(x,y)$ degenerated for all $x,y \in X$. Let the mapping $S : X \to X$ satisfies the condition*

$$d(S^n x,S^n y) \precsim {}_{i_2} Ad(x,y) + B\frac{d(x,S^n x)\,d(y,S^n y)}{1+d(x,y)} \\ +C[d(x,S^n x)+d(y,S^n y)] + D[d(x,S^n y)+d(y,S^n x)]$$

for all $x,y \in X$ where A,B,C and D are non negative real numbers such that $A+\sqrt{2}B+2C+2sD<1$. Then S and T have a unique common fixed point in X.

Proof: By Corollary 10 there exists a unique point $u \in X$ such that

$$S^n u = u.$$

Therefore,

$$\begin{aligned} d(Su,u) &= d(SS^n u,S^n u) = d(S^n Su,S^n u) \\ &\precsim i_2 Ad(Su,u) + B\frac{d(Su,S^n Su)\,d(u,S^n u)}{1+d(Su,u)} \\ &\qquad +C[d(Su,S^n Su)+d(u,S^n u)] + D[d(Su,S^n u)+d(u,S^n Su)] \\ &\precsim i_2 Ad(Su,u) + B\frac{d(Su,Su)\,d(u,u)}{1+d(Su,u)} \\ &\qquad +C[d(Su,Su)+d(u,u)] + D[d(Su,u)+d(u,Su)] \\ &\precsim I_2(A+2D)\,d(Su,u) \end{aligned}$$

$$\Rightarrow \|d(Su,u)\| \leq (A+2D)\|d(Su,u)\|$$
$$\Rightarrow \|d(Su,u)\| = 0$$
$$\Rightarrow Su = u.$$

This completes the proof of the corollary.

References

Agarwal, R., Goswami, M.P. and **Agarwal, R. P.** (2014) : Convolution theorem and applications of bicomplex Laplace transform, *Adv. Math. Sci. Appl.*, **24**(1), 113.

Agarwal, R., Goswami, M.P. and **Agarwal, R.P.** (2015) : Tauberian theorem and applications of bicomplex Laplace-Stieltjes transform, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms*, **22**, 141.

Agarwal, R., Goswami, M.P. and **Agarwal, R.P.** (2014) : Bicomplex version of Stieltjes transform and applications, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms*, **21**, 229.

Ahmad, J., Azam, A. and **Saejung, S.** (2014) : Common fixed point results for contractive mappings in complex valued metric spaces, **Fixed point Theory and App.**, 1.

Alpay, D., Luna-Elizarrarás, M.E., Shapiro, M. and **Struppa, D.C.** (2014) : Basics of functional analysis with bicomplex scalars and bicomplex schur analysis, Springer Briefs.

Azam, A., Brain, F. and **Khan, M.** (2011) : Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, **32** (3), 243.

Banerjee, A. Datta, S.K. and **Hoque, A.** (2014) : Inverse Laplace transform for bicomplex variables, *Math. Inverse Probl.*, **1**(1), 8.

Bhatt, S., Chaukiyal, S. and **Dimri, R. C.** (2011) : A common fixed point theorem for weakly compatible maps in complex valued metric spaces, *Internat. J. Math. Sci. Appl.*, **1**(3), 1385.

Bhatt, S., Chaukiyal, S. and **Dimri, R. C.** (2011) : Common fixed point of mappings satisfying rational inequality in complex valued metric space, *Internat. J. Pure. App. Math.*, **73**(2), 159.

Charak, K.S., Kumar, R. and **Rochon, D.** (2013) : Infinite dimensional bicomplex spectral decomposition theorem, *Adv. Appl. Clifford Algebras*, **23**, 593.

Choi, J., Datta, S.K., Biwswas, T. and **Islam, N.** (2017) : Some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. *Honam Mathematical J.*, **39**(1), 115.

Choudhury, B.S., Metiya, N. and **Konar, P.** (2015) : Fixed point results in partially ordered complex valued metric spaces for rational type expressions, *Bangmod Int. J. Math. & Comp. Sci.* , **1**(1), 55.

Choudhury, B. S., Metiya, N. and **Konar, P.** (2015) : Fixed point results for rational type construction in partially ordered complex valued metric spaces, *Bulletin Internat. Math. Virtual Institute.* **5**, 73.

Colombo, F., Sabadini, I., Struppa, D.C., Vajiac, A. and **Vajiac, M.** (2010) : Singularities of functions of one and several bicomplex variables, Ark. Math..

Datta, S.K. and **Ali, S.** (2017) : Some common fixed point theorems for two weakly compatible mappings in complex valued metric spaces, *Thai J. Math.*, **15**(3), 797.

Datta, S.K., Pal, D., Biswas, N. and **Sarkar, S.** (2020) : On the study of fixed point theorems in bicomplex valued metric spaces, *Journal of Calcutta Mathematical Society*, **16**(1), 73.

Datta, S.K. and **Pal, D.** (2020) : Some study on fixed point theorems in bicomplex valued metric spaces, LAP Lambert Academic Publishing.

De, H., Bie, Struppa, D.C., Vajiac, A. and **Vajiac, M.** (2012) : The Cauchy-Kowalevski product for bicomplex holomorphic functions, *Math. Nachr.* **285**(10), 1230.

Dubey, A.K., Shukla, R. and **Dubey, R.P.** (2015) : Some common fixed point theorems for contractive mappings in complex-valued b-metric spaces, *Asian J. Math. Appl.*, **2015**, Article ID ama0266.

Goyal, R. (2007) : Bicomplex polygamma function, *Tokyo J. Math.*, **30**(2), 523.

Jaishree (2012) : On conjugates and modulii of bicomplex numbers, *Internat. J. Engrg. Sci. Tech.*, **4**(6), 2567.

Jebril, I.H., Datta, S.K., Sarkar, R. and **Biswas, N.** (2019) : Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces, *Journal of Interdisciplinary Mathematics*, **22**(7), 1071.

Kumar, A. and **Kumar, P.** (2011) : Bicomplex version of Laplace transform, *Internat. J. Engrg. Tech.*, **3**(3), 225.

Kumar, A., Kumar, P. and **Dixit, P.** (2011) : Maximum and minimum modulus principle for bicomplex holomorphic functions, *Internat. J. Engrg. Tech.*, **3**(2), 1484.

Lavoie, R.G., Marchildon, L. and **Rochon, D.** (2011) : Hilbert space of the bicomplex quantum harmonic oscillator, *AIP Conf. Proc.*, **1327**, 148.

Lavoie, R.G., Marchildon, L. and **Rochon, D.** (2011) : Finite-dimensional bicomplex Hilbert spaces, *Adv. Appl. Clifford Algebras*, **21**(3), 561.

Lavoie, R.G., Marchildon, L. and **Rochon, D.** (2010) : Infinite-dimensional bicomplex Hilbert spaces, *Ann. Funct. Anal.*, **1**(2), 75.

Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D.C. and **Vajiac, A.** (2013) : Complex Laplacian and derivatives of bicomplex functions, *Complex Anal.Oper. Theory*, **7**(5), 1675.

Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D.C. and **Vajiac, A.** (2012) : Bicomplex numbers and their elementary functions, *Cubo*, **14**(2), 61.

Mukheimer, A.A. (2014) : Some common fixed point theorems in complex-valued b-metric spaces, *Sci. World J.*, **2014**, Article ID 587825.

Price, G. B. (1991) : An introduction to multicomplex spaces and functions, Marcel Dekker, New York.

Rao, K.P.R., Ramga Swamy. P. and **Rajendra Prasad, J.** (2013) : A common fixed point theorem in complex-valued b-metric spaces, *Bull. Math. & Stat. Res.*, **1**(1), 1.

Rochon, D. (2004) : A bicomplex Riemann zeta function, *Tokyo J. Math.*, 27(2), 357.

Rochon, D. and **Shapiro, M.** (2004) : On algebraic properties of bicomplex and hyperbolic numbers, *An. Univ. Oradea Fasc. Mat.*, **11**, 71.

Rouzkard, F. and **Imdad, M.** (2012) : Some common fixed point theorems on complex valued metric spaces, *Computers and Math. App.* **64**, 1866.

Scorza Dragoni, G. (1934) : Sulle funzioni olomorfe di una variabile bicomplessa, *Reale Accad. d'Italia, Mem. Classe Sci. Nat. Fis. Mat.*, **5**, 597.

Segre, C. (1892) : Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Math. Ann.*, **40**, 413.

Sintunavarat, W. and **Kumam, P.** (2011) : Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric spaces, *J. Appl. Math.*, **2011**:14, Article ID 637958.

Spampinato, N. (1935) : Estensione nel campo bicomplesso di due teoremi, del Levi-Civita e del Severi, per le funzioni olomorfe di due variablili bicomplesse I, II, *Reale Accad. Naz. Lincei*, **22**(6), 38, 96.

Spampinato, N. (1936) : Sulla rappresentazione delle funzioni do variabile bicomplessa totalmente derivabili, *Ann. Mat. Pura Appl.*, **14**(4), 305.

Tripathy, B.C., Paul, S. and **Das, N. R.** (2013) : Banach's and Kannan's fixed point results in fuzzy 2-metric spaces; *Proyecciones J. Math.*, **32**(4), 359.

Tripathy, B.C., Paul, S. and **Das, N.R.** (2014) : A fixed point theorem in a generalized fuzzy metric space; *Boletim da Sociedade Paranaense de Matemática,* **32**(2), 221.

Tiwari, R. and **Shukla, D.P.** (2012) : Six maps with a common fixed point in complex valued metric spaces, *Res. J. Pure Algebra.*, **2**(12), 365.

Verma, R.K. and **Pathak, H.K.** (2013) : Common fixed point theorems for a pair of mappings in complex-valued metric spaces, *J. Math. Sci. & Comp. Sci.*, **6**, 18.

[1,4]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
P.O. : KALYANI, DIST. : NADIA, PIN : 741235
WEST BENGAL, INDIA
E-MAIL : [1]sanjibdatta05@gmail.com, [4]jayantas324@gmail.com

[2]DEPARTMENT OF MATHEMATICS
PROF. SYED NURUL HASAN COLLEGE
P.O. : FARAKKA BARRAGE, DIST. : MURSHIDABAD, PIN : 742212
WEST BENGAL, INDIA
E-MAIL : dpal.math@gmail.com

[3]DEPARTMENT OF MATHEMTICS
GOUR MAHAVIDYALAYA
P.O. : MANGALBARI, DIST. : MALDA, PIN : 732142
WEST BENGAL, INDIA
E-MAIL : rakeshsarkar.malda@gmail.com