# SOME COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED METRIC SPACES UNDER BOTH RATIONAL TYPE CONTRACTION AND COUPLED FIXED POINT MAPPINGS

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# SOME COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED METRIC SPACES UNDER BOTH RATIONAL TYPE CONTRACTION AND COUPLED FIXED POINT MAPPINGS

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ABSTRACT. In the paper we investigate some common fixed point theorems for a pair of mappings satisfying certain rational type contraction condition and having a unique common coupled fixed point in the framework of bicomplex valued metric spaces . Our results are the generalizations of coupled fixed point theorems of Bhatt et. al. [3] and Savitri et. al.[13].

## 1. INTRODUCTION AND PRELIMINARIES

We write the set of real and complex numbers respectively as $\mathbb{C}_0$ and $\mathbb{C}_1$. Azam et. al. [1] introduced the concept of complex valued metric spaces. Bhatt et al.[3] proved some common fixed point theorems satisfying rational inequality and Datta et. al.[5] proved some common fixed point theorems under the contractive condition in $\mathbb{C}_1$ . The concept of the coupled fixed point was first introduced by Bhaskar & Laxikantham[2]. Kang et.al.[9] also proved some coupled fixed point theorems in $\mathbb{C}_1$. Recently Savitri et.al.[13] investigated some coupled fixed point theorems in $\mathbb{C}_1$. There are several numbers of generalization such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, $D$-metric spaces, cone metric spaces, and bicomplex valued metric spaces. The space $\mathbb{C}_2$ of bicomplex numbers and bicomplex valued metric spaces are most important to our work. Ronn [10] made an extensive use of complex pairs to develop the algebras of quaternions and bicomplex numbers. Although both the algebras generalize the complex algebra, the commutativity of the multiplication operation is lost in quaternion algebra. Segre[12] treated an algebra whose elements were bicomplex numbers. Elena et.al. [6] showed how to introduce elementary functions, such as polynomials, exponential and trigonometric functions in the algebra of bicomplex numbers as well as their inverses which is not possible in the case of quaternions incidentally. They also showed how these elementary functions enjoy the properties that are very similar to those enjoyed by their complex counterparts. Very recently Choi et. al. [4] investigated some fixed point theorems in connection with two weakly compatible mappings in $\mathbb{C}_2$. Also Jebril et. al.{[7]& [8]} investigated some common fixed point theorems under rational contraction for a pair of mappings in $\mathbb{C}_2$.

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The partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied

$(i)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, $(ii)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$, $(iii)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$, $(iv)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of $(ii)$, $(iii)$ and $(iv)$ is satisfied and we write $z_1 \prec z_2$ if only $(iv)$ is satisfied. Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}_1$ is defined as follows

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$ ;

(2) If $z_1 \precsim z_2, z_2 \precsim z_3$ then $z_1 \precsim z_3$ and

(3) If $z_1 \precsim z_2$ and $\lambda < 1$ is a non-negative real number then $\lambda z_1 \precsim z_2$.

Azam et. al. [1] defined the complex valued metric space in the following way.

**Definition 1.1.** [1] Let $X$ be a non empty set and the mapping $d : X \times X \to \mathbb{C}_1$, satisfies the following conditions:

$(d_1)$ $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$(d_2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(d_3)$ $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a complex valued metric on $X$ and $(X, d)$ is called a complex valued metric space.

The space $\mathbb{C}_2$ is the first in an infinite sequence of multicomplex spaces which are generalizations of $\mathbb{C}_1$.

The notion of the space $\mathbb{C}_2$ was defined by Segre[12] as

$$\mathbb{C}_2 = \{w : w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3, p_k \in \mathbb{C}_0,\ 0 \leq k \leq 3\}$$

$$\text{i.e., } \mathbb{C}_2 = \{w = z_1 + i_2 z_2 \mid z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = p_0 + i_1 p_1,\ z_2 = p_2 + i_1 p_3$ and $i_1$, $i_2$ are independent imaginary units such that $i_1^2 = -1 = i_2$. The product of $i_1$ and $i_2$ defines a hyperbolic unit $j$ such that $j^2 = 1$. The product of all units are commutative and satisfy

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

**Definition 1.2.** For a bicomplex number $w = z_1 + i_2 z_2$, the norm is denoted by $\|z_1 + i_2 z_2\|$ and defined by

$$\|z_1 + i_2 z_2\| = \left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}} = \left(|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2\right)^{\frac{1}{2}}.$$

If we take $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$ for $p_k \in \mathbb{C}_0,\ k = 0, 1, 2, 3$ then the norm of $w$ is defined by

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}}.$$

The partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ was defined by Choi et. al. [4] as $u \precsim_{i_2} v$ if and only if $u_1 \precsim u_2$ and $v_1 \precsim v_2$, where $u_1, u_2, v_1, v_2 \in \mathbb{C}$. The bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ on a non-empty set $X$ and the structure $(X, d)$ on $\mathbb{C}_2$ were defined by Choi et. al. [4] accordingly.

By the deduction of Rochon & Shapiro [11] we get the results

$(i)$ $\|uv\| \leq \sqrt{2}\,\|u\|\,\|v\|$ for any $u$, $v \in \mathbb{C}_2$ ; .

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$(ii)$ $\|uv\| = \|u\|\,\|v\|$ for any $u,\, v \in \mathbb{C}_2$ with at least one of them is degenerated;
$(iii)$ $\left\|\frac{1}{u}\right\| = \frac{1}{\|u\|}$ for any degenerated bicomplex number $u$ with $0 \precsim_{i_2} u$.

**Definition 1.3.** The max function for the partial order $\precsim_{i_2}$ on $\mathbb{C}_2$ is defined as follows

$$(i)\ \max\{u, v\} = v \Leftrightarrow u \precsim_{i_2} v;$$

$$(ii)\ u \precsim_{i_2} \max\{u, v\} \Longrightarrow u \precsim_{i_2} v \text{ or } \|u\| \leq \|v\|;$$

$$(iii)\ \max\{u, v\} = v \Longleftrightarrow u \precsim_{i_2} v \text{ or } \|u\| \leq \|v\|.$$

**Definition 1.4.** Let $\{z_n\}$ be an arbitrary sequence in $\mathbb{C}_2$. Then $\{z_n\}$ is said to be Cauchy in $\mathbb{C}_2$ if and only if $d\{z_n, z_{n+m}\} \prec c$ for all $0 \prec c \in \mathbb{C}_2$ as $n \to \infty$.

**Definition 1.5.** Let $\{z_n\}$ be any Cauchy sequence in $\mathbb{C}_2$. Then $\mathbb{C}_2$ is said to be complete if every Cauchy sequence $\{z_n\}$ is converges to a point $z_0$ in $\mathbb{C}_2$.

**Definition 1.6.** [9] An element $(z, z') \in X \times X$ is called a coupled fixed point of the mapping $S : X \times X \to X$ if

$$S(z, z') = z \text{ and } S(z', z) = z'.$$

**Lemma 1.1.** [1] Let $(X, d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is Cauchy if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Bhatt et. al.[3] investigated the common fixed point of mappings satisfying rational inequality in $\mathbb{C}_1$ and obtained the following result.

**Theorem 1.1.** [3] Let $(X, d)$ be a complex valued metric space. Let the mappings $S, T : X \to X$ be satisfy

$$d\left(Sx, Ty\right) \leq \frac{a\left[d\left(x, Sx\right) d\left(x, Ty\right) + d\left(y, Ty\right) d\left(y, Sx\right)\right]}{d\left(x, Ty\right) + d\left(y, Sx\right)},$$

for all $x, y \in X$, where $0 \leq a < 1$. Then $S$ and $T$ have a unique common fixed point.

Recently Savitri et. al.[13] introduced the concept of coupled fixed point in $\mathbb{C}_1$ and obtained the following theorem.

**Theorem 1.2.** [13] Let $(X, d)$ be a complete complex valued metric space and $S, T : X^2 \to X$ be the mappings satisfy

$$d\left(S\left(x_1, x_2\right), T\left(y_1, y_2\right)\right)$$
$$\leq \alpha \frac{\left[d\left(x_1, S\left(x_1, x_2\right)\right) d\left(x_1, T\left(y_1, y_2\right)\right) + d\left(x_1, T\left(y_1, y_2\right)\right) d\left(x_1, S\left(x_1, x_2\right)\right)\right]}{d\left(x_1, T\left(y_1, y_2\right)\right) d\left(x_1, S\left(x_1, x_2\right)\right)}$$

for all $x_1, x_2, y_1, y_2 \in X$, where $0 \leq \alpha < 1$. Then $S$ and $T$ have a unique common coupled fixed point.

Our works are the generalization of the above two theorems of Bhatt et. al. [3] and Savitri et.al.[13] and some extensions of these theorems. The purpose of this paper is also to introduce the concepts of Choi et. al.[4] and Jebril et. al.[7] to improve the above results in $\mathbb{C}_2$. In fact we wish to explore and also extend the relevant results of $\mathbb{C}_1$ in $\mathbb{C}_2$.

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## 2. Main Result

In this section we prove some theorems and corollaries followed by a lemma, an effective tool of our works.

**Lemma 2.1.** Let $(X, d)$ be a bicomplex valued metric space and $\{z_n\}$ be a sequence in $\mathbb{C}_2$. Then $\{z_n\}$ is Cauchy if and only if $\|d(z_n, z_{n+m})\| \to 0$ as $n \to \infty$.

*Proof.* We take $n + m = n'$. Then $n' \to \infty$ as $n \to \infty$ for any positive integer $m$. Also we take $z_n = z_{n_1} + i_2 z_{n_2}$ and $z_{n'} = z_{n'_1} + i_2 z_{n'_2}$, where $z_{n_1}, z_{n_2}, z_{n'_1}, z_{n'_2} \in \mathbb{C}_1$. Now

$$\begin{aligned}
\|d(z_n, z_{n+m})\| &= \|d(z_n, z_{n'})\| = \|z_n - z_{n'}\| \\
&= \|(z_{n_1} + i_2 z_{n_2}) - (z_{n'} + i_2 z_{n'})\| \\
&= \left\|(z_{n_1} - z_{n'_1}) + i_2(z_{n_2} - z_{n'_2})\right\|
\end{aligned}$$

$$= \left(\left|z_{n_1} - z_{n'_1}\right|^2 + \left|z_{n_2} - z_{n'_2}\right|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

Therefore from Equation (2.1) we get that the sequence $\{z_n\}$ is Cauchy in $\mathbb{C}_2$ if and only if the sequences $\{z_{n_1}\}$ and $\{z_{n_2}\}$ are Cauchy in $\mathbb{C}_1$. Again by Lemma(1.1) we have that the sequences $\{z_{n_1}\}$ and $\{z_{n_2}\}$ are Cauchy in $\mathbb{C}_1$ if and only if

$$|d(z_{n_1}, z_{n_1+m_1})| \to 0 \text{ and } |d(z_{n_2}, z_{n_2+m_2})| \to 0,$$

for some $m_1$ and $m_2$ and $n_1, n_2 \to \infty$. Therefore we get that

$$|z_{n_1} - z_{n_1+m_1}| \to 0 \text{ and } |z_{n_2} - z_{n_2+m_2}| \to 0, \tag{2.2}$$

for some $m_1$ and $m_2$ and $n_1, n_2 \to \infty$. Also from (2.1) & (2.2) we may conclude that $\{z_n\}$ is Cauchy in $\mathbb{C}_2$ if and only if $\|d(z_n, z_{n+m})\| \to 0$ as $n \to \infty$.

This completes the proof of the lemma. □

**Theorem 2.1.** Let $(X, d)$ be a bicomplex valued metric space and $S, T : X \to X$ be two mappings satisfying the condition

$$d(Sz, Tz') \precsim_{i_2} \frac{\alpha\left[(z, Sz) \cdot d(z, Tz') + d(z', Tz') \cdot d(z', Sz)\right]}{d(z, Tz') + d(z', Sz)} \tag{2.3}$$

for all $z, z' \in X$ and $0 \leq \alpha < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.

*Proof.* Let $z_0$ be any arbitrary point in $X$. We construct a sequence $\{z_n\}$ in the following way

$$Sz_n = z_{n+1} \text{ and } Tz_{n+1} = z_{n+2},\ n = 0, 1, 2, 3, \cdots$$

then we have

$$\begin{aligned}
& d(z_{n+1}, z_{n+2}) = d(Sz_n, Tz_{n+1}) \\
\precsim_{i_2} & \frac{\alpha\left[d(z_n, Sz_n) \cdot d(z_n, Tz_{n+1}) + d(z_{n+1}, Tz_{n+1}) \cdot d(z_{n+1}, Sz_n)\right]}{d(z_n, Tz_{n+1}) + d(z_{n+1}, Sz_n)} \\
= & \frac{\alpha\left[d(z_n, z_{n+1}) \cdot d(z_n, z_{n+2}) + d(z_{n+1}, z_{n+2}) \cdot d(z_{n+1}, z_{n+1})\right]}{d(z_n, z_{n+2}) + d(z_{n+1}, z_{n+1})} \\
\precsim_{i_2} & \frac{\alpha\left[d(z_n, z_{n+1}) \cdot d(z_n, z_{n+2})\right]}{d(z_n, z_{n+2})} \precsim_{i_2} \alpha d(z_n, z_{n+1}).
\end{aligned}$$

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Consequently for $n \geq 0$, we obtain that

$$d(z_{n+1}, z_{n+2}) \leq \alpha d(z_n, z_{n+1}) \leq \alpha^2 d(z_{n-1}, z_n) \leq ... \leq \alpha^{n+1} d(z_0, z_1).$$

Therefore for any $m > n$ we get that

$$\begin{aligned} d(z_m, z_n) &\precsim_{i_2} d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + ... + d(z_{m-1}, z_m) \\ &\precsim_{i_2} \alpha^n d(z_0, z_1) + \alpha^{n+1} d(z_0, z_1) + ... + \alpha^{m-1} d(z_0, z_1) \\ &= \{\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1}\} d(z_0, z_1) = \frac{\alpha^n}{1-\alpha} d(z_0, z_1), \end{aligned}$$

which implies that

$$\|d(z_m, z_n)\| \leq \frac{\alpha^n}{1-\alpha} \|d(z_0, z_1)\|. \quad (2.4)$$

Now on taking $n \to \infty$ we get that $\frac{\alpha^n}{1-\alpha} \to 0$ and hence $\|d(z_m, z_n)\| \to 0$. Therefore using Lemma 2.1 we can say that $\{z_n\}$ is Cauchy in $X$. Again since $X$ is a complete therefore the sequence $\{z_n\}$ converges to a point $u$ in $X$. Now we have to show that $Su = u$. If not, then $d(Su, u) = p$ (say) for some $p > 0$. So using (2.3 ) we get

$$\begin{aligned} p &\precsim_{i_2} d(u, z_{n+2}) + d(z_{n+2}, Su) = d(u, z_{n+2}) + d(Su, Tz_{n+1}) \\ &\precsim_{i_2} d(u, z_{n+2}) + \frac{\alpha[d(u, Su) d(u, Tz_{n+1}) + d(z_{n+1}, Tz_{n+1}) d(z_{n+1}, Su)]}{d(u, Tz_{n+1}) + d(z_{n+1}, Su)} \\ &\precsim_{i_2} d(u, z_{n+2}) + \frac{\alpha[p.d(u, z_{n+2}) + d(z_{n+1}, z_{n+2}) d(z_{n+1}, Su)]}{d(u, z_{n+2}) + d(z_{n+1}, Su)}, \end{aligned}$$

which implies that

$$\|p\| \leq \|d(u, z_{n+2})\| + \frac{\alpha\sqrt{2}[\|p\| \cdot \|d(u, z_{n+2})\| + \|d(z_{n+1}, z_{n+2})\| \, \|d(z_{n+1}, Su)\|]}{\|d(u, z_{n+2}) + d(z_{n+1}, Su)\|}. \quad (2.5)$$

Taking limit as $n \to \infty$ we get that $\|d(u, z_{n+2})\| \to 0$ and $\|d(z_{n+1}, z_{n+2})\| \to 0$. Also we have

$$\alpha[\|p\| \cdot \|d(u, z_{n+2})\| + \sqrt{2} \|d(z_{n+1}, z_{n+2})\| \, \|d(z_{n+1}, Su)\|] \to 0 \text{ as } n \to \infty.$$

So (2.5) implies that $\|p\| \leq 0$, a contradiction. Therefore $\|p\| = 0$ and hence $Su = u$. Similarly we obtain that $Tu = u$. Hence $S$ and $T$ have a common fixed point in $X$.

**Uniqueness:**

Now we show that $S$ and $T$ have a unique common fixed point. For this let us assume that $u^*$ be another common fixed point of $S$ and $T$. Then we have

$$d(Su, Tu^*) \precsim_{i_2} \frac{\alpha\left[(u, Su) \cdot d(u, Tu^*) + d(u^*, Tu^*) \cdot d(u^*, Su)\right]}{d(u, Tu^*) + d(u^*, Su)},$$

which implies that $\|d(Su, Tu^*)\| \leq 0$ and hence $u = u^*$.

Thus the proof of the theorem is established. □

**Corollary 2.1.** Let $(X, d)$ be a bicomplex valued metric space and $T : X \to X$ be the mapping satisfying the condition

$$d(T^n z, T^n z') \precsim_{i_2} \frac{[d(z, T^n z) d(z, T^n z') + d(z', T^n z') d(z', T^n z)]}{d(z, T^n z') + d(z', T^n z)}$$

for all $z, z' \in X$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}$. Then $T$ has a unique common fixed point in $X$.

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*Proof.* Setting $S = T$ in (2.3) and using the result (2.5) inductively we may easily prove the corollary. □

**Theorem 2.2.** Let $(X, d)$ be a complete bicomplex valued metric space and the mappings $S, T : X \to X$ be two mappings satisfying

$$d(Sz, Tz') \precsim_{i_2} \alpha d(z, z') + \beta \max \left\{ d(z, z'), \frac{\beta d(z, Sz)\, d(z', Tz')}{1 + d(Sz, Tz')} \right\}, \tag{2.6}$$

for all $z, z' \in X$ such that $z \neq z'$and $\alpha, \beta$ are nonnegative reals with $\alpha + \beta < 1$. Then $S$ and $T$ have a unique common fixed point.

*Proof.* Let $z_0$ be an arbitrary point in $X$ and define $z_{2k+1} = Sz_{2k}$, $z_{2k+2} = Tz_{2k+1}$, $k = 0, 1, 2, 3, \cdots$. Then we have

$$\begin{aligned} d(z_{2k+1}, z_{2k+2}) &= d(Sz_{2k}, Tz_{2k+1}) \\ &\precsim_{i_2} \alpha d(z_{2k}, z_{2k+1}) + \beta \max \left\{ d(z_{2k}, z_{2k+1}), \frac{d(z_{2k}, Sz_{2k})\, d(z_{2k+1}, Tz_{2k+1})}{1 + d(Sz_{2k}, Tz_{2k+1})} \right\} \\ &\precsim_{i_2} \alpha d(z_{2k}, z_{2k+1}) + \beta \max \left\{ d(z_{2k}, z_{2k+1}), \frac{d(z_{2k}, z_{2k+1})\, d(z_{2k+1}, z_{2k+2})}{1 + d(z_{2k+1}, z_{2k+2})} \right\}. \end{aligned} \tag{2.7}$$

Again we know that $d(z_{2k+1}, z_{2k+2}) < 1 + d(z_{2k+1}, z_{2k+2})$. Therefore we have

$$\frac{d(z_{2k+1}, z_{2k+2})}{1 + d(z_{2k+1}, z_{2k+2})} < 1,$$

which yields that

$$\max \left\{ d(z_{2k}, z_{2k+1}), \frac{d(z_{2k}, z_{2k+1})\, d(z_{2k+1}, z_{2k+2})}{1 + d(z_{2k+1}, z_{2k+2})} \right\} = d(z_{2k}, z_{2k+1}). \tag{2.8}$$

Therefore from(2.7) & (2.8) we obtain that

$$d(z_{2k+1}, z_{2k+2}) \precsim \alpha d(z_{2k}, z_{2k+1}) + \beta d(z_{2k}, z_{2k+1}) \precsim (\alpha + \beta) d(z_{2k}, z_{2k+1}). \tag{2.9}$$

Now taking norm on both side of (2.9), we have

$$\| d(z_{2k+1}, z_{2k+2}) \| \leq (\alpha + \beta) \| d(z_{2k}, z_{2k+1}) \|. \tag{2.10}$$

Similarly we get that

$$\begin{aligned} d(z_{2k+2}, z_{2k+3}) &= d(Tz_{2k+1}, Sz_{2k+2}) = d(Sz_{2k+2}, Tz_{2k+1}) \\ &\precsim_{i_2} \alpha d(z_{2k+2}, z_{2k+1}) + \beta \max \left\{ d(z_{2k+2}, z_{2k+1}), \frac{d(z_{2k+2}, Sz_{2k+2})\, d(z_{2k+1}, Tz_{2k+1})}{1 + d(Sz_{2k+2}, Tz_{2k+1})} \right\} \\ &\precsim_{i_2} \alpha d(z_{2k+2}, z_{2k+1}) + \beta \max \left\{ d(z_{2k+2}, z_{2k+1}), \frac{d(z_{2k+2}, z_{2k+3})\, d(z_{2k+1}, z_{2k+2})}{1 + d(z_{2k+3}, z_{2k+2})} \right\}. \end{aligned} \tag{2.11}$$

Also we have $d(z_{2k+2}, z_{2k+3}) < 1 + d(z_{2k+2}, z_{2k+3})$.

$$\text{i.e., } \frac{d(z_{2k+2}, z_{2k+3})}{1 + d(z_{2k+2}, z_{2k+3})} < 1,$$

which implies that

$$\max \left\{ d(z_{2k+2}, z_{2k+1}), \frac{d(z_{2k+2}, z_{2k+3})\, d(z_{2k+1}, z_{2k+2})}{1 + d(z_{2k+3}, z_{2k+2})} \right\} = d(z_{2k+2}, z_{2k+1}). \tag{2.12}$$

Therefore by (2.11) & (2.12) we obtain that

$$||d(z_{2k+3}, z_{2k+2})|| \leq (\alpha + \beta) ||d(z_{2k+2}, z_{2k+1})||$$

$$\text{i.e., } ||d(z_{2k+2}, z_{2k+3})|| \leq (\alpha + \beta) ||d(z_{2k+1}, z_{2k+2})|| . \tag{2.13}$$

So by (2.10) & (2.13) and using $\delta = \alpha + \beta < 1$ we have

$$||d(z_{n+1}, z_{n+2})|| \leq \delta ||d(z_n, z_{n+1})|| \leq +... + \delta^{n+1} ||d(z_0, z_1)|| \text{ for all } n \in \mathbb{N}.$$

Also for any $m > n$, we obtain that

$$\begin{aligned} ||d(z_n, z_{n+m})|| &\leq ||d(z_n, z_{n+1})|| + ||d(z_{n+1}, z_{n+2})|| + ... + ||d(z_{n+m-1}, z_{n+m})|| \\ &\leq \left|\delta^n + \delta^{n+1} + ... + \delta^{n+m-1}\right| ||d(z_0, z_1)|| \\ &\leq \frac{\delta^n}{1-\delta} ||d(z_0, z_1)|| . \end{aligned}$$

Hence $||d(z_n, z_{n+m})|| \leq \frac{\delta^n}{1-\delta} ||d(z_0, z_1)|| \to 0$ as $n \to \infty$ for an $m(> n) \in \mathbb{N}$. Therefore $\{z_n\}$ is Cauchy. Since $\mathbb{C}_2$ is complete, there exists some $u \in \mathbb{C}_2$ such that $z_n \to u$ as $n \to \infty$. Thus we get

$$\lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} Tz_{2n+1} = u = Su.$$

If possible let $u \neq Su$. Then $d(u, Su) = p > 0$. Now we can write

$$p \precsim_{i_2} d(u, z_{2k+2}) + d(z_{2k+2}, Su) = d(u, z_{2k+2}) + d(Tz_{2k+1}, Su)$$

$$\precsim_{i_2} d(u, z_{2k+2}) + \alpha d(z_{2k+1}, u) + \beta \max\left\{d(z_{2k+1}, u), \frac{d(u, Su)d(z_{2k+1}, Tz_{2k+1})}{1 + d(Tz_{2k+1}, Su)}\right\}$$

$$= d(u, z_{2k+2}) + \alpha d(z_{2k+1}, u) + \beta \max\left\{d(z_{2k+1}, u), \frac{d(u, Su)d(z_{2k+1}, z_{2k+2})}{1 + d(z_{2k+2}, Su)}\right\},$$

which implies that

$$||p|| \leq ||d(u, z_{2k+2})|| + \alpha ||d(z_{2k+1}, u)||$$

$$+\beta \max\left\{\|d(z_{2k+1}, u)\|, \frac{\sqrt{2}\,\|d(u, Su)\|\,\|d(z_{2k+1}, z_{2k+2})\|}{\|1 + d(z_{2k+2}, Su)\|}\right\}.$$

Therefore by taking limit as $n \to \infty$ we get $||p|| \leq 0$, a contradiction. So we have $u = Su$. Similarly we can show that $u = Tu$. Therefore $u$ is a common fixed point of $S$ and $T$.

**Uniqueness:**

To prove the uniqueness of common fixed point of $S$ and $T$, let us assume that $u^* \in X$ be another common fixed point of $S$ and $T$. Then we have $Su^* = Tu^* = u^*$. Now

$$d(u, u^*) \quad = \quad d(Su, Tu^*) \precsim_{i_2} \alpha d(u, u^*) + \beta \max\left\{d(u, u^*), \frac{d(u, Su)d(u^*, Tu^*)}{1 + d(Su, Tu^*)}\right\},$$

implies that

$$d(u, u^*) \precsim_{i_2} \alpha d(u, u^*) + \beta d(u, u^*) = (\alpha + \beta)d(u, u^*).$$

Therefore we obtain that

$$||d(u, u^*)|| \leq \delta ||d(u, u^*)|| ,$$

with $\delta < 1$, a contradiction. Therefore we get $u^* = u$. This proves the uniqueness of common fixed point.

Hence the proof of the theorem is established. □

**Corollary 2.2.** Let $(X, d)$ be a complete bicomplex valued metric space and $T : X \to X$ be the mapping satisfying

$$d(Tz, Tz') \precsim_{i_2} \alpha d(z, z') + \beta \max \left\{ d(z, z'), \frac{\beta d(z, Tz)\, d(z', Tz')}{1 + d(Tz, Tz')} \right\},$$

for all $z, z' \in X$ such that $z \neq z'$and $\alpha, \beta$ are non negative real numbers with $\alpha+ \beta < 1$. Then $T$ has a unique common fixed point in $X$.

*Proof.* Setting $S = T$ in (2.6) we may easily prove the result. □

**Theorem 2.3.** Let $(X, d)$ be a bicomplex valued metric space and $S, T : X \times X \to X$ be two mappings satisfying

$$d(S(z_{1,}z_2), T(z_1', z_2')) \precsim_{i_2} \alpha \frac{[d(z_1, S(z_1, z_2))d(z_1, T(z_1', z_2')) + d(z_1', T(z_1', z_2'))d(z_1', S(z_1, z_2))]}{d(z_1, T(z_1', z_2')) + d(z_1', S(z_1, z_2))},$$

for all $z_1, z_2, z_1', z_2' \in X$ and $0 \leq \alpha < 1$. Then $S$ and $T$ have a unique coupled fixed point.

*Proof.* We construct the sequences $\{z_n\}$ and $\{z_n'\}$ in $X$ for any two arbitrary point $z_0, z_0' \in X$ such a way that $z_{n+1} = S(z_n, z_n'), z_{n+2} = T(z_{n+1}, z_{n+1}')$ and $z_{n+1}' = S(z_n', z_n), z_{n+2}' = T(z_{n+1}', z_{n+1})$, for all $n = 0, 1, 2, \cdots$. Now

$$d(z_{n+1}, z_{n+2}) = d(S(z_n, z_n'), T(z_{n+1}, z_{n+1}'))$$

$$\precsim_{i_2} \frac{\alpha[d(z_n, S(z_n, z_n')) \cdot d(z_n, T(z_{n+1}, z_{n+1}')) + d(z_{n+1}, T(z_{n+1}, z_{n+1}')) \cdot d(z_{n+1}, S(z_n, z_n'))]}{d(z_n, T(z_{n+1}, z_{n+1}')) + d(z_{n+1}, S(z_n, z_n'))}$$

$$= \frac{\alpha[d(z_n, z_{n+1}) \cdot d(z_n, z_{n+2}) + d(z_{n+1}, z_{n+2})d(z_{n+1}, z_{n+1})]}{d(z_n, z_{n+2}) + d(z_{n+1}, z_{n+1})} = \alpha d(z_n, z_{n+1}), \text{ for all } n \geq 0.$$

Hence we have

$$d(z_{n+1}, z_{n++2}) \precsim_{i_2} \alpha d(z_n, z_{n+1}) \precsim_{i_2} \alpha^2 d(z_{n-1}, z_n) \precsim_{i_2} ... \precsim_{i_2} \alpha^{n+1} d(z_0, z_1).$$

Now for $m > n$, we obtain that

$$\begin{aligned} d(z_n, z_m) &\precsim_{i_2} d(z_n, z_{n+1}) + d(z_{n+1}, z_{n++2}) + ... + d(z_{m-1}, z_m) \\ &= \alpha^n d(z_0, z_1) + \alpha^{n+1} d(z_0, z_1) + ... + \alpha^{m-1} d(z_0, z_1) \\ &= \alpha^n d(z_0, z_1)\{1 + \alpha + \alpha^2 + ... + \alpha^{m-n-1}\} \\ &= \frac{\alpha^n}{1 - \alpha} d(z_0, z_1), \end{aligned}$$

implying that

$$\|d(z_n, z_m)\| \leq \frac{\alpha^n}{1 - \alpha} \|d(z_0, z_1)\|.$$

So we have $\|d(z_n, z_m)\| \to 0$ as $n, m \to \infty$. Hence $\{z_n\}$ is a Cauchy sequence. Also

$$d(z_{n+1}', z_{n+2}') = d(S(z_n', z_n), T(z_{n+1}', z_{n+1}))$$

$$\precsim_{i_2} \frac{\alpha[d(z'_n, S(z'_n, z_n)) \cdot d(z'_n, T(z'_{n+1}, z_{n+1})) + d(z'_{n+1}, T(z'_{n+1}, z_{n+1}))d(z'_{n+1}, S(z'_n, z_n))]}{d(z'_n, T(z'_{n+1}, z_{n+1})) + d(z'_{n+1}, S(z'_n, z_n))}$$
$$= \frac{\alpha[d(z'_n, z'_{n+1}) \cdot d(z'_n, z'_{n+2}) + d(z'_{n+1}, z'_{n+2})d(z'_{n+1}, z'_{n+1})]}{d(z'_n, z'_{n+2}) + d(z'_{n+1}, z'_{n+1})} \precsim_{i_2} \alpha d(z'_n, z'_{n+1}),$$

which implies that

$$d(z'_{n+1}, z'_{n+2}) \precsim_{i_2} \alpha d(z'_n, z'_{n+1}), \text{ for all } n \geq 0.$$

Hence we get that

$$d(z'_{n+1}, z'_{n++2}) \precsim_{i_2} \alpha d(z'_n, z'_{n+1}) \precsim_{i_2} \alpha^2 d(z'_{n-1}, z'_n) \precsim_{i_2} ... \precsim_{i_2} \alpha^{n+1} d(z'_0, z'_1).$$

Now for $m > n$, we have

$$\begin{aligned} d(z'_n, z'_m) &\precsim_{i_2} d(z'_n, z'_{n+1}) + d(z'_{n+1}, z'_{n++2}) + ... + d(z'_{m-1}, z'_m) \\ &= \alpha^n d(z'_0, z'_1) + \alpha^{n+1} d(z'_0, z'_1) + ... + \alpha^{m-1} d(z'_0, z'_1) \\ &= \alpha^n d(z'_0, z'_1)\{1 + \alpha + \alpha^2 + ...\alpha^{m-n-1}\} \\ &= \frac{\alpha^n}{1-\alpha} d(z'_0, z'_1). \end{aligned}$$

Therefore we have $\|d(z'_n, z'_m)\| \leq \frac{\alpha^n}{1-\alpha} \|d(z'_0, z'_1)\|$. i.e., $\|d(z'_n, z'_m)\| \to 0$ as $n, m \to \infty$. This implies that $\{z'_n\}$ is also Cauchy in $X$. So $\{z_n\}$ and $\{z'_n\}$ are Cauchy in $X$. Again since $X$ is complete, $\exists\, z, z' \in X$ such that $\{z_n\}$ and $\{z'_n\}$ converges to $z, z'$ respectively. Now we will show that $S(z, z') = z$. If not then $\exists\, \bar{z} \in X$ such that $d(S(z, z'), z) = \bar{z}$ where $\|\bar{z}\| > 0$. Now

$$\bar{z} \precsim_{i_2} d(z, z_{n+2}) + d(z_{n+2}, S(z, z')) = d(z, z_{n+2}) + d(S(z, z'), T(z_{n+1}, z'_{n+1}))$$
$$\precsim_{i_2} d(z, z_{n+2}) + \alpha \frac{[d(z, S(z, z'))d(z, T(z_{n+1}, z'_{n+1})) + d((z_{n+1}, T(z_{n+1}, z'_{n+1})d(z_{n+1}, S(z, z'))]}{d(z, T(z_{n+1}, z_{n+1})) + d(z_{n+1}, S(z, z'))}$$
$$= d(z, z_{n+2}) + \alpha \frac{[\bar{z} \cdot d(z, z_{n+2}) + d((z_{n+1}, z_{n+2})d(z_{n+1}, S(z, z'))]}{d(z, z_{n+2}) + d(z_{n+1}, S(z, z'))},$$

which implies that

$$\|\bar{z}\| \leq \|d(z, z_{n+2})\| + \sqrt{2}\alpha \frac{[\|\bar{z}\| \, \|d(z, z_{n+2})\| + \|d((z_{n+1}, z_{n+2})\| \, \|d(z_{n+1}, S(z, z'))\|]}{\|d(z, z_{n+2}) + d(z_{n+1}, S(z, z'))\|}.$$

On taking limit as $n \to \infty$ we get $\|\bar{z}\| \leq 0$, a contradiction. So $\|\bar{z}\| = 0$ and $S(z, z') = z$. Also if possible, let $d(S(z', z), z') = t$ with $\|t\| > 0$. Then we have

$$t \precsim_{i_2} d(z', z'_{n+2}) + d(z'_{n+2}, S(z', z)) = d(z', z'_{n+2}) + d(S(z', z)T(z'_{n+1}, z_{n+1})$$
$$\precsim_{i_2} d(z', z'_{n+2}) + \alpha \frac{[d(z', S(z', z))d(z', T(z'_{n+1}, z_{n+1}) + d(z'_{n+1}, T(z'_{n+1}, z_{n+1}))d(z'_{n+1}, S(z', z))]}{d(z', T(z'_{n+1}, z_{n+1})) + d(z'_{n+1}, S(z', z))}$$
$$= d(z', z'_{n+2}) + \alpha \frac{[td(z', z'_{n+2}) + d(z'_{n+1}, z'_{n+2})d(z'_{n+1}, S(z', z))]}{d(z', z'_{n+2}) + d(z'_{n+1}, S(z', z))},$$

which implies that

$$\|t\| \leq \left\|d(z', z'_{n+2})\right\| + \sqrt{2}\alpha \frac{[\|t\| \left\|d(z', z'_{n+2})\right\| + \left\|d(z'_{n+1}, z'_{n+2})\right\| \left\|d(z'_{n+1}, S(z', z))\right\|]}{\left\|d(z', z'_{n+2}) + d(z'_{n+1}, S(z', z))\right\|}.$$

On taking limit as $n \to \infty$ we get that $\|t\| < 0$, a contradiction. Therefore $t = 0$ and $S(z', z) = z'$. Similarly, we may show that $T(z, z') = z$ and $T(z', z) = z'$. Hence $(z, z')$ is the common coupled fixed point of $S$ and $T$.

**Uniqueness:**

For uniqueness, let us suppose that $(u, v)$ be another common coupled fixed point of $S$ and $T$. Then

$$d(z,u) = d(S(z,z'),T(u,v)) \precsim_{i_2} \frac{[d(z,S(z,z'))d(z,T(u,v)) + d(u,T(u,v))d(u,S(z,z'))]}{d(z,T(u,v)) + d(u,S(z,z'))}.$$

Therefore we get $\| d(z,u)\| \leq 0$ i.e., $z = u$. Similarly, we may show that $z' = v$. So $(z, z')$ is the unique common coupled fixed point for $S$ and $T$.

This completes the proof of the theorem. □

## 3. Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrices under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

## References


[1] A. Azam, B. Fisher and M. Khan, *Common Fixed Point Theorems in Complex Valued Metric Spaces,* Num. Func. Anal. Opt. 32, (2011), pp. 243 − 253.

[2] T. G. Bhaskar and V. Laxikantam, Fixed point theorem in partially ordered metric spaces and applications, Nonlinear Analysis, 65, (2006),pp.1379 − 1393.

[3] S. Bhatt, S. Chaukiyal and R. C. Dimri, Common fixed point of mappings satisfying rational inequality in complex valued metric space, Int. J. Pure Appl. Math., 73(2), (2011), pp.159 − 164.

[4] J. Choi, S. K. Datta, T. Biswas and N. Islam, Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces, Honam Mathematical J., 39 ,(2017), pp.115 − 126.

[5] S. K. Datta and S. Ali, A Common Fixed Point Result in Complex Valued Metric Spaces under Contractive Condition, International Journal of Advanced Scientific and Technical Research, 6(2),(2012), pp. 467 − 475.

[6] M. Elena, M. Shapiro and D.C.Strupra, Bicomplex Number andTheir Elementary Functions, CUBO A Mathematics Journal, 14(2), (2012) pp. 61 − 80.

[7] I. H. Jebril, S. K. Datta, R. Sarkar and N. Biswas, Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces, Journal of Interdisciplinary Mathematics, 22(7), (2019), pp. 1071 − 1082, DOI:10.1080/09720502.2019.1709318.

[8] I. H. Jebril, S. K. Datta, R. Sarkar and N. Biswas, Common fixed point theorems under rational contractions using two mappings and six mappings and coupled fixed point theorem in bicomlex valued b-metric space, TWMS J. of Apl. & Eng. Math. (Accepted).

[9] S. M. Kang and M. Kumar, Coupled Fixed Point Theorems in Complex Valued Metric Spaces, Int. Journal of Math. Analysis, 7(46), (2013), pp. 2269 − 2277.

[10] S. Ronn, Bicomplex algebra and function theory, math.CV/0101200.

[11] D. Rochon and M. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, Anal. Univ. Oradea, fasc. math, 11(2004), pp. 71 − 110.

[12] C. Segre, Le Rappresentazioni Reali delle Forme Complesse a Gli Enti Iperalgebrici, Math. Ann., 40, (1892), pp. 413 − 467.

[13] Savitri and N. Hooda, A Common Coupled Fixed Point Theorem in Complex Valued Metric Space, Int. J. Comput. Appl., 109(4), (2015), pp. 10 − 12.

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