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COMMON FIXED POINT THEOREMS FOR THREE SELF MAPPINGS IN BICOMPLEX VALUED METRIC SPACES

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Abstract: In the paper we prove some common fixed point theorems for three self mappings in a bicomplex valued metric spaces. Our results generalize the literature due to Azam *et al.* (2011) & Sintunavarat & Kumam (2012) by using both the ideas of two weakly compatible mappings and rational contractions for a pair of mappings in bicomplex valued metric space.

Keywords and Phrases: Bicomplex valued metric space, compatible mappings, common fixed point, weakly compatible mapping, expansive metric space.

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1. Introduction and Preliminaries

During the last fifty years, fixed point theories in complex valued metric spaces are emerging areas of works in the field of the complex as well as functional analysis. Banach's fixed point theorem plays a major role in the fixed point theory. It has applications in many branches of mathematics. The famous Banach's theorem states that "Let (X, d) be a metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $(0, 1)$. Then T has a unique fixed point $x^* \in X$." Azam *et al.* [2] made a generalization by introducing a complex valued metric space using some contractive type conditions. Sintunavarat *et al.* [9] generalized this result by replacing the constants of contraction by some control functions.

Already there have been a number of generalizations of metric spaces such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D -metric spaces and cone metric spaces. Recently, Mahanta *et al.* [7] proved a common fixed point result for three self mapping in complex valued metric spaces which generalized the results of [2] and [9].

In 1892, Segre [8] made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this significant work of Segre failed to earn the attention of the mathematicians for almost over a century. Also the area of research work related to bicomplex valued metric was empty upto few years ago. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology. Bicomlex valued metric space [3] is a very recent generalization of metric space. Choi *et al.* [3] proved some common fixed point theorems under rational contractions for a pair of weakly compatible mappings in bicomplex valued metric spaces. Recently Jebril *et al.* [6] have proved some common fixed point theorems under rational contractions using two and six mappings respectively in bicomlex valued b-metric space. The aim of this paper is to introduce three mappings in bicomplex valued metric space to obtain a different type of results.

We write regular complex number as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Let $\mathbb{C}$ be the set of complex numbers and z_1 and $z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re\,(z_2) \text{ and } Im\,(z_1) \leq Im\,(z_2)\,.$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied

(*i*) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, (*ii*) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$, (*iii*) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and (*iv*) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$, i.e., one of (*ii*), (*iii*) and (*iv*) is satisfied further, we write $z_1 \prec z_2$ if only (iv) is satisfied. Taking these into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}$ are deduced as follows.

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$;

(2) If $z_1 \precsim z_2,\ z_2 \precsim z_3$ then $z_1 \precsim z_3$ and

(3) If $z_1 \precsim z_2$ and λ, μ be two real numbers such that $0 < \lambda < \mu$, then $\lambda z_1 \precsim \mu z_2$.

The notion of the set of bicomplex numbers was defined by Segre [8] as

$$\mathbb{C}_2 = \{w : w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3, p_k \in \mathbb{R}, 0 \leq k \leq 3\}$$

$$\text{i.e., } \mathbb{C}_2 = \{w = z_1 + i_2 z_2 \mid z_1, z_2 \in \mathbb{C}\}.$$

For a bicomplex number $w = z_1 + i_2 z_2$, the norm is denoted as $\|w = z_1 + i_2 z_2\|$ and is defined in the following manner.

$$\begin{aligned} \|w\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}}. \end{aligned}$$

When $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$ for $p_k \in \mathbb{R}, k = 0, 1, 2, 3$, then

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}}.$$

The partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ has been defined by Choi *et al.* [3], as $u \precsim_{i_2} v$ if and only if $u_1 \precsim u_2$ and $v_1 \precsim v_2$ where $u_1, u_2, v_1, v_2 \in \mathbb{C}$.

Also Choi *et al.* [3]defined the bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ on a non-empty set X and bicomplex valued metric space (X, d) in $\mathbb{C}_2$ as follows.

Definition 1.1. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim_{i_2} d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a bicomplex valued metric on X and (X, d) is called the bicomplex valued metric space.

Example 1.1. Consider $X = \mathbb{C}$ and a mapping $d : X \times X \to \mathbb{C}_2$ defined by $d(z_1, z_2) = i_2 \|z_1 - z_2\|, z_1, z_2 \in X$ where $\|\ \ \|$ is the complex modulus. One can easily check that (X, d) is a bicomplex valued metric space.

Definition 1.2. [4] *Let S and T be self mappings of a set X. If $w = Tz = Sz$*

for some z in X, then z is called a coincidence point of S and T and w is called a point of coincidence of T and S.

We are giving the following definitions in $\mathbb{C}_2$ which will be very fruitful to our results.

Definition 1.3. *The* max *function for the partial order $\precsim_{i_2}$ on $\mathbb{C}_2$ is defined as follows*

$$(i)\ \max\{u,v\} = v \Leftrightarrow u \precsim_{i_2} v;$$

$$(ii)\ u \precsim_{i_2} \max\{u,v\} \Longrightarrow u \precsim_{i_2} v \text{ or } \|u\| \leq \|v\|;$$

$$(iii)\ \max\{u,v\} = v \Longleftrightarrow u \precsim_{i_2} v \text{ or } \|u\| \leq \|v\|.$$

Definition 1.4. *Let $\{z_n\}$ be a sequence in $\mathbb{C}_2$ and $z \in \mathbb{C}_2$. If for every $c \in \mathbb{C}_2$ with $0 \prec_2 c$ there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(z_n, z) \prec_2 c$, then z is called the limit of $\{z_n\}$ and we write $\lim_{n\to\infty} z_n = z$ or, $z_n \to z$ as $n \to \infty$.*

Definition 1.5. *Let (X,d) be a metric space in $\mathbb{C}_2$ and let $\{z_n\}$ be a sequence in $\mathbb{C}_2$. Then $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}_2$ if and only if $\|d(z_n, z_m)\| \to 0$ as $n, m \to \infty$.*

Definition 1.6. *Let $\{z_n\}$ be any Cauchy sequence in $\mathbb{C}_2$. Then $\mathbb{C}_2$ is said to be complete if every Cauchy sequence $\{z_n\}$ converges to a point z_0 in $\mathbb{C}_2$.*

Definition 1.7. *Two self-maps S and T in $\mathbb{C}_2$ are said to be weakly compatible if $STz = TSz$ whenever $Sz = Tz$, i.e., they commute at their coincidence points.*

Definition 1.8. *A mapping $T : X \to X$ in a bicomplex valued metric space (X,d) is said to be expansive if there is a real constant $c > 1$ satisfying $cd(z,z') \precsim_{i_2} d(Tz,Tz')$, for all $z, z' \in X$.*

Mahanta *et al.* [7] have investigated common fixed points of three self mappings in complex valued metric spaces and obtained the following theorem.

Theorem 1.1. [7] *Let (X,d) be a complex valued metric space and $f, S, T : X \to X$. Suppose there exit mappings $\Lambda_1, \Lambda_2 : X \to [0,1)$ such that for all $x, y \in X$:*

(i) $\Lambda_i(Sx) \leq \Lambda_i(fx)$ and $\Lambda_i(Tx) \leq \Lambda_i(fx)$ for $i = 1,2$;

(ii) $\Lambda_1(fx) + \Lambda_2(fx) < 1$;

(iii) $d(Sx,Ty) \preceq \Lambda_1(fx)\, d(fx,fy) + \frac{\Lambda_1(fx) d(fx,Sx) d(fy,Ty)}{1+d(fx,fy)}$.

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is complete, then f, S and T have a unique point of coincidence. Moreover, if (S,f) and (T,f) are weakly compatible, then f, S and T have a unique common fixed point in X.

In this paper, we prove some common fixed point results for three self mappings in a bicomplex valued metric spaces. We generalize the literature of Azam *et al.*

[2], Sintunavarat & Kumam [9] and modify Theorem 1.1 of [7] by using the results of two weakly compatible mappings in $\mathbb{C}_2$ (see [3], [5]).

2. Main Results

In this section, we prove some common fixed point theorems on bicomplex valued metric space and some corollaries of this theorem and an example to justify the results.

Lemma 2.1. [1] *Let X be a non empty set and mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then S, T and f have a unique fixed point.*

Theorem 2.1. *Let (X, d) be a bicomplex valued metric space and $S, T, f : \mathbb{C}_2 \to \mathbb{C}_2$. Also suppose there exist mappings $\alpha_1, \alpha_2 : \mathbb{C}_2 \to [0, 1)$ such that for all $z, z' \in X$*

(i) $\alpha_i(Sz) \leq \alpha_i(fz)$ and $\alpha_i(Tz) \leq \alpha_i(fz)$ for $i = 1, 2$;

(ii) $\alpha_1(fz) + \alpha_2(fz) < 1$ and

(iii) $d(Sz, Tz') \precsim_{i_2} \alpha_1(fz)d(fz, fz') + \frac{\alpha_2(fz)d(fz,Sz)d(fz',Tz')}{1+d(fz,fz')}$.

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is complete, then f, S and T have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible, then f, S and T have a unique common fixed point in $\mathbb{C}_2$.

Proof. Let z_0 be arbitrary point in $\mathbb{C}_2$. Choose a point $z_1 \in \mathbb{C}_2$ such that $fz_1 = Sz_0$ which is possible since $S(X) \subseteq f(X)$. Also we may choose a point $z_2 \in \mathbb{C}_2$ satisfying $fz_2 = Tz_1$ since $S(X) \subseteq f(X)$. Continuing in this way, we can construct a sequence $\{fz_n\}$ in $f(\mathbb{C}_2)$ such that

$$f(z_n) = \begin{cases} Sz_{n-1}, \text{if } n \text{ is odd} \\ Tz_{n-1}, \text{if } n \text{ is even.} \end{cases}$$

If $n \in \mathbb{N}$ is odd, then by using hypothesis we obtain that

$$\begin{aligned} d(fz_n, fz_{n+1}) &= d(Sz_{n-1}, Tz_n) \\ &\precsim_{i_2} \alpha_1(fz_{n-1})d(fz_{n-1}, fz_n) + \frac{\alpha_2(fz_{n-1})d(fz_{n-1}, Sz_{n-1})d(fz_n, Tz_n)}{1 + d(fz_{n-1}, fz_n)} \\ &= \alpha_1(fz_{n-1})d(fz_{n-1}, fz_n) + \frac{\alpha_2(fz_{n-1})d(fz_{n-1}, fz_n)d(fz_n, fz_{n+1})}{1 + d(fz_{n-1}, fz_n)}, \end{aligned}$$

which implies that

$$\begin{aligned} &\|d(fz_n, fz_{n+1})\| \\ &\leq \alpha_1(fz_{n-1}) \|d(fz_{n-1}, fz_n)\| + \alpha_2(fz_{n-1}) \|d(fz_n, fz_{n+1})\| \frac{\|d(fz_{n-1}, fz_n)\|}{\|1 + d(fz_{n-1}, fz_n)\|}. \end{aligned} \tag{2.1}$$

Again we know that $\|d(fz_{n-1}, fz_n)\| \leq \|1 + d(fz_{n-1}, fz_n)\|$, which shows that

$$\frac{\|d(fz_{n-1}, fz_n)\|}{\|1 + d(fz_{n-1}, fz_n)\|} < 1. \tag{2.2}$$

Therefore by (2.1) and condition (i), we get that

$$\begin{aligned}
\|d(fz_n, fz_{n+1})\| &\leq \alpha_1(fz_{n-1})\|d(fz_{n-1}, fz_n)\| + \alpha_2(fz_{n-1})\|d(fz_n, fz_{n+1})\| \\
&= \alpha_1(Tz_{n-2})\|d(fz_{n-1}, fz_n)\| + \alpha_2(Tz_{n-2})\|d(fz_n, fz_{n+1})\| \\
&\leq \alpha_1(fz_{n-2})\|d(fz_{n-1}, fz_n)\| + \alpha_2(fz_{n-2})\|d(fz_n, fz_{n+1})\| \\
&= \alpha_1(Sz_{n-3})\|d(fz_{n-1}, fz_n)\| + \alpha_2(Sz_{n-3})\|d(fz_n, fz_{n+1})\| \\
&\leq \alpha_1(fz_{n-3})\|d(fz_{n-1}, fz_n)\| + \alpha_2(fz_{n-3})\|d(fz_n, fz_{n+1})\| \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \alpha_1(fz_0)\|d(fz_{n-1}, fz_n)\| + \alpha_2(fz_0)\|d(fz_n, fz_{n+1})\|,
\end{aligned}$$

which implies that

$$\|d(fz_n, fz_{n+1})\| \leq \frac{\alpha_1(fz_0)}{1 - \alpha_2(fz_0)}\|d(fz_{n-1}, fz_n)\|.$$

If $n \in \mathbb{N}$ is even, then

$$d(fz_n, fz_{n+1}) = d(Tz_{n-1}Sz_n) = d(Sz_n, Tz_{n-1})$$

$$\begin{aligned}
&\precsim_{i_2} \alpha_1(fz_n)d(fz_n, fz_{n-1}) + \frac{\alpha_2(fz_n)d(fz_n, Sz_n)d(fz_{n-1}, Tz_{n-1})}{1 + d(fz_n, fz_{n-1})} \\
&= \alpha_1(fz_n)d(fz_n, fz_{n-1}) + \frac{\alpha_2(fz_n)d(fz_n, fz_{n+1})d(fz_{n-1}, fz_n)}{1 + d(fz_n, fz_{n-1})}.
\end{aligned}$$

Therefore by (2.2) and above inequality, we obtain that

$$\begin{aligned}
\|d(fz_n, fz_{n+1})\| &\leq \alpha_1(fz_n)\|d(fz_n, fz_{n-1})\| \\
&\quad + \alpha_2(fz_n)\|d(fz_n, fz_{n+1})\| \frac{\|d(fz_{n-1}, fz_n)\|}{\|1 + d(fz_n, fz_{n-1})\|}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_1(fz_n)\,\|d(fz_n,fz_{n-1})\| + \alpha_2(fz_n)\,\|d(fz_n,fz_{n+1})\| \\
&= \alpha_1(Tz_{n-1})\,\|d(fz_n,fz_{n-1})\| + \alpha_2(Tz_{n-1})\,\|d(fz_n,fz_{n+1})\| \\
&\leq \alpha_1(fz_{n-1})\,\|d(fz_n,fz_{n-1})\| + \alpha_2(fz_{n-1})\,\|d(fz_n,fz_{n+1})\| \\
&= \alpha_1(Sz_{n-2})\,\|d(fz_n,fz_{n-1})\| + \alpha_2(Sz_{n-2})\,\|d(fz_n,fz_{n+1})\| \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \alpha_1(fz_0)\,\|d(fz_n,fz_{n-1})\| + \alpha_2(fz_0)\,\|d(fz_n,fz_{n+1})\|,
\end{aligned}$$

which gives that

$$\|d(fz_n,fz_{n+1})\| \leq \frac{\alpha_1(fz_0)}{1-\alpha_2(fz_0)}\,\|d(fz_{n-1},fz_n)\|.$$

Thus for any positive integer n, it must be the case that

$$\|d(fz_n,fz_{n+1})\| \leq \frac{\alpha_1(fz_0)}{1-\alpha_2(fz_0)}\,\|d(fz_{n-1},fz_n)\|. \tag{2.3}$$

If we set $\lambda = \frac{\alpha_1(fz_0)}{1-\alpha_2(fz_0)}$ then $\lambda < 1$, since by (ii) we have $\frac{\alpha_1(fz_0)}{1-\alpha_2(fz_0)} < 1$.

Therefore by repeated application of (2.3), we get that

$$\begin{aligned}
\|d(fz_n,fz_{n+1})\| &\leq \lambda\,\|d(fz_{n-1},fz_n)\| \\
&\leq \lambda^2\,\|d(fz_{n-1},fz_n)\| \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \lambda^n\,\|d(fz_{n-1},fz_n)\|.
\end{aligned}$$

Now, for all $m,n \in \mathbb{N}$, $m > n$, we have

$$d(fz_n,fz_m) \precsim_{i_2} d(fz_n,fz_{n+1}) + d(fz_{n+1},fz_{n+2}) + ... + d(fz_{m-1},fz_m),$$

which implies that

$$\begin{aligned}
\|d(fz_n,fz_m)\| &\leq \|d(fz_n,fz_{n+1})\| + \|d(fz_{n+1},fz_{n+2})\| + ... + \|d(fz_{m-1},fz_m)\| \\
&\leq \lambda^n\,\|d(fz_0,fz_1)\| + \lambda^{n+1}\,\|d(fz_0,fz_1)\| + ... \\
&\quad +\lambda^{m-1}\,\|d(fz_0,fz_1)\| \\
&= (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1})\,\|d(fz_0,fz_1)\| \\
&\leq \frac{\lambda^n}{1-\lambda}\,\|d(fz_0,fz_1)\|.
\end{aligned}$$

Since $\lambda < 1$, taking limit as $m, n \to \infty$, we get $\|d(fz_n, fz_m)\| \to 0$ which implies that $\{fz_n\}$ is a Cauchy sequence in $f(\mathbb{C}_2)$. Again by completeness of $f(\mathbb{C}_2)$, there exist $u, v \in \mathbb{C}$ such that $fz_n \to v = fu$. Now

$$\begin{aligned}
&d(fu, Tu) \precsim d(fu, fz_{2n+1}) + d(fz_{2n+1}, Tu)\\
&= d(fu, fz_{2n+1}) + d(Sz_{2n}, Tu)\\
&\precsim_{i_2} d(fu, fz_{2n+1}) + \alpha_1(fz_{2n})d(fz_{2n}, fu)\\
&+\frac{\alpha_2(fz_{2n})d(fz_{2n}, Sz_{2n})d(fu, Tu)}{1 + d(fz_{2n}, fu)}.
\end{aligned}$$

Since $1 \precsim 1 + d(fz_{2n}, fu)$, from above inequality we have

$$\begin{aligned}
\|d(fu, Tu)\| &\leq \|d(fu, fz_{2n+1})\| + \alpha_1(fz_{2n})\|d(fz_{2n}, fu)\|\\
&\quad+\frac{\alpha_2(fz_{2n})\|d(fz_{2n}, Sz_{2n})\|\,\|d(fu, Tu)\|}{\|1 + d(fz_{2n}, fu)\|}\\
&\leq \|d(fu, fz_{2n+1})\| + \alpha_1(fz_{2n})\|d(fz_{2n}, fu)\|\\
&\quad+\alpha_2(fz_{2n})\|d(fz_{2n}, Sz_{2n})\|\,\|d(fu, Tu)\|.
\end{aligned}$$

Therefore, we get that

$$\begin{aligned}
\|d(fu, Tu)\| &\leq \|d(fu, fz_{2n+1})\| + \alpha_1(fz_{2n})\|d(fz_{2n}, fu)\|\\
&\quad+\alpha_2(fz_{2n})\|d(fz_{2n}, fz_{2n+1})\|\,\|d(fu, Tu)\|.
\end{aligned}$$

Taking limit as $n \to \infty$, it follows that $\|d(fu, Tu)\| = 0$ and hence $d(fu, Tu) = 0$. Therefore, $fu = Tu = v$. Similarly, we can show that $fu = Su = v$. Thus, $fu = Su = Tu = v$ and v becomes a common point of coincidence of f, S and T.

For uniqueness, assume that there exists another point $w(\neq v) \in \mathbb{C}_2$ such that $fz = Sz = Tz = w$ for some $z \in \mathbb{C}_2$. Thus,

$$\begin{aligned}
&d(v, w) = d(Su, Tz)\\
&\precsim_{i_2} \alpha_1(fu)d(fu, fz) + \frac{\alpha_2(fu)d(fu, Su)d(fz, Tz)}{1 + d(fu, fz)}\\
&= \alpha_1(v)d(v, w) + \frac{\alpha_2(v)d(v, v)d(w, w)}{1 + d(v, w)}\\
&= \alpha_1(v)d(v, w),
\end{aligned}$$

which implies that

$$\|d(v, w)\| \leq \alpha_1(v)\|d(v, w)\|.$$

Since $0 \leq \alpha_1(v) < 1$, it follows that $\|d(v,w)\| = 0$ and so $v = w$. If (S,f) and (T,f) are weakly compatible, then by Lemma 2.1 f, S and T have a unique common fixed point in $\mathbb{C}_2$.

This completes the proof of the theorem.

Corollary 2.1. *Let (X,d) be a complete bicomplex valued metric space and $S,T : \mathbb{C}_2 \to \mathbb{C}_2$. Suppose there exist mappings $\alpha_1, \alpha_2 : \mathbb{C}_2 \to [0,1)$ such that for all $z, z' \in \mathbb{C}_2$*

(i) $\alpha_i(Sz) \leq \alpha_i(z)$ and $\alpha_i(Tz) \leq \alpha_i(z)$ for $i = 1,2$;

(ii) $\alpha_1(z) + \alpha_2(z) < 1$ and

(iii) $d(Sz,Tz') \precsim_{i_2} \alpha_1(z)d(z,z') + \frac{\alpha_2(z)d(z,Sz)d(z',Tz')}{1+d(z,z')}$.

Then S and T have a unique common fixed point in $\mathbb{C}_2$.

Proof. The result follows from Theorem 2.1 by taking $f = I$, the identity mapping on $\mathbb{C}_2$.

Corollary 2.2. *Let (X,d) be a complete bicomplex valued metric space and $S,T : \mathbb{C}_2 \to \mathbb{C}_2$. If S and T satisfy*

$$d(Sz,Tz') \leq \lambda d(z,z') + \frac{\mu d(z,Sz)d(z',Tz')}{1+d(z,z')},$$

for all $z, z' \in \mathbb{C}_2$, where λ, μ are nonnegative real numbers with $\lambda + \mu < 1$, then S and T have a unique common fixed point.

Proof. The desired result can be obtained from Theorem 2.1 by setting $\alpha_1(z) = \lambda, \alpha_2(z) = \mu$ and $f = I$.

Corollary 2.3. *Let (X,d) be a bicomplex valued metric space and $f,T : \mathbb{C}_2 \to \mathbb{C}_2$ be such that $T(X) \subseteq f(X)$ and $f(X)$ is complete. Suppose there exist mappings $\alpha_1, \alpha_2 : \mathbb{C}_2 \to [0,1)$ such that for all $z, z' \in \mathbb{C}_2$*

(i) $\alpha_i(Tz) \leq \alpha_i(fz)$ for $i = 1,2$;

(ii) $\alpha_1(fz) + \alpha_2(fz) < 1$ and

(iii) $d(Tz,Tz') \precsim_{i_2} \alpha_1(fz)d(fz,fz') + \frac{\alpha_2(fz)d(fz,Tz)d(fz',Tz')}{1+d(fz,fz')}$.

Then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point in $\mathbb{C}_2$.

Proof. The conclusion of the corollary follows from Theorem 2.1, by considering $S = T$.

Corollary 2.4. *Let (X,d) be a bicomplex valued metric space and $f,T : \mathbb{C}_2 \to \mathbb{C}_2$ satisfy*

$$d(Tz,Tz') \precsim_{i_2} \lambda d(fz,fz') + \frac{\mu d(fz,Tz)d(fz',Tz')}{1+d(fz,fz')},$$

for all $z, z' \in \mathbb{C}_2$, *where* λ, μ *are nonnegative real numbers with* $\lambda+\mu < 1$. *If* $T(X) \subseteq f(X)$ *and* $f(X)$ *is complete, then* f *and* T *have a unique point of coincidence. Moreover, if* f *and* T *are weakly compatible, then* f *and* T *have a unique common fixed point in* $\mathbb{C}_2$.

Proof. Putting $S = T, \alpha_1(z) = \lambda$ and $\alpha_2(z) = \mu$ in Theorem 2.1 we can prove this result.

Similarly the following corollary is obtained by considering $S = T$ and $f = I$ in conditions $(i), (ii) \& (iii)$ of Theorem 2.1.

Corollary 2.5. *Let* (X, d) *be a complete bicomplex valued metric space and* $T : \mathbb{C}_2 \to \mathbb{C}_2$. *Suppose there exist mappings* $\alpha_1, \alpha_2 : \mathbb{C}_2 \to [0, 1)$ *such that for all* $z, z' \in \mathbb{C}_2$

(i) $\alpha_i(Tz) \leq \alpha_i(z)$ *for* $i = 1, 2$;

(ii) $\alpha_1(z) + \alpha_2(z) < 1$ *and*

(iii) $d(Tz, Tz') \precsim_{i_2} \alpha_1(z)d(z, z') + \frac{\alpha_2(z)d(z,Tz)d(z',Tz')}{1+d(z,z')}$.

Then T *has a unique fixed point in* $\mathbb{C}_2$.

Corollary 2.6. *Let* (X, d) *be a complete bicomplex valued metric space and* $f, T : \mathbb{C}_2 \to \mathbb{C}_2$ *satisfy*

$d(Tz, Tz') \precsim_{i_2} \lambda d(fz, fz') + \frac{\mu d(fz,Tz)d(fz',Tz')}{1+d(fz,fz')}$.

for all $z, z' \in X(\subseteq \mathbb{C}_2)$, *where* λ, μ *are nonnegative reals with* $\lambda + \mu < 1$. *If* $T(X) \subseteq f(X)$ *and is complete, then* f *and* T *have a unique point of coincidence. Moreover, if* f *and* T *are weakly compatible, then* f *and* T *have a unique common fixed point in* X.

Example 2.1. Let $X = [1, \infty)$. Define $T, f : X \to X$ by $Tx = a_1x + a_2$ and $fx = a_3x + a_4$ where $a_1 < a_2 < a_3 < a_4$ in X. If d_u is the usual metric on X , then T and f are not the contraction mappings on X as for all $x, y \in X$, $d_u(Tx, Ty) = a_1 |x - y|$ and $d_u(fx, fy) = a_3 |x - y|$.

So, we can not apply Banach contraction theorem to find the unique fixed point 1 of T and f.

We consider a bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ by

$$d(x, y) = |x - y| + i_1 |x - y| + i_2 |x - y| + i_1i_2 |x - y|$$

Then (X, d) is a complete bicomplex valued metric space.

Now, $d(Tx, Ty) = a_1 [|x - y| + i_1 |x - y| + i_2 |x - y| + i_1i_2 |x - y|]$

$= \frac{a_1}{a_3} d(fx, fy) \leq hd(fx, fy)$, where $0 < h = \frac{a_1}{a_3} < 1$.

Since $T(X) = f(X) = X$,we have all the conditions of Corollary 2.6 with $\lambda = h, \mu = 0$. So applying Corollary 2.6, we can obtain a unique fixed point 1 of T

and f in X.

Theorem 2.2. *Let (X, d) be a complete bicomplex valued metric space and let $f : \mathbb{C}_2 \to \mathbb{C}_2$ be an onto expansive mapping, i.e., $f(\mathbb{C}_2) = \mathbb{C}_2$ and there exists a real constant $c > 1$ such that $cd(z, z') \precsim_{i_2} d(fz, fz')$ for all $z, z' \in \mathbb{C}_2$ then f has a unique fixed point in $\mathbb{C}_2$.*

Proof. In Corollary 2.4, we see that $\lambda + \mu < 1$. If we put $\mu = 0$, then $\lambda < 1$ which implies $\frac{1}{\lambda} > 1$. Therefore setting $\mu = 0$ and $c = \frac{1}{\lambda}$ in Corollary 2.4, we get that

$$cd(Tz, Tz') \precsim_{i_2} d(fz, fz'). \tag{2.4}$$

Now if we put $T = I$ in (2.4), then we get the required result. Thus the proof of the theorem is established.

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