Some Common Fixed Point Theorems for Rational Type Contraction Mappings in Partial Metric Spaces

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Abstract

In the paper we investigate some common fixed point theorems for rational type contraction mappings in partial metric spaces. Our results are the extension of the result of Karapinar & Yuksel[6] and modification of the results of Jebril et al.[3].
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1 Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in Mathematics and it has a broad number applications in different branches of Mathematics. In this theory contraction is one of the main tools to prove the existence and uniqueness of a fixed point. The Banach contraction principle {cf.[2]} is a very popular and effective tool to solving existence problems in many branches of mathematical analysis and it is an active area of research since 1922. The famous Banach theorem {cf. [2]} states that " *Let* (X, d) *be a complete metric space and and* T *be a mapping of* X *into itself satisfying* $d(Tx, Ty) \leq kd(x, y) \forall\ x, y \in X$, *where* k *is a constant in* $(0, 1)$. *Then* T *has a unique fixed point* $x^* \in X$."

Partial metric spaces as introduced by Matthews {[8]&[9]}, are the generalization of the notion of metric spaces in which the condition $d(x, x) = 0$ is replaced by the condition $d(x, x) \leq d(x, y)$ in the definition of metric. Different approaches in this area have been reported including applications of mathematical techniques to computer science {cf. [7]&[13]}. In [9], Matthews discussed some properties of convergence of sequences and investigated the fixed point theorems for contractive mapping on partial metric spaces. Any mapping T of a complete partial metric space X into itself that satisfies the inequality $d(Tx, Ty) \leq kd(x, y)$, where $0 \leq k < 1$, for all $x, y \in X$, has a unique fixed point. Recently, many authors {cf. [10],[11][12], [13] &[14]} have focused on this subject and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces. The definition of partial metric space is given by Matthews in [8] as follows.

Definition 1.1. *[8]Let* X *be a non empty set and let* $p : X \times X \to \mathbb{R}_0^+$ *satisfy*

$$(PM1).\ \ x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y),$$

$$(PM2).\quad p(x,x) \le p(x,y),$$
$$(PM3).\quad p(x,y) = p(y,x),$$
$$(PM4).\quad p(x,y) \le p(x,z) + p(z,y) - p(z,z),$$

for all $x, y, z \in X$, where $\mathbb{R}_0^+ = [0,\infty)$. Then the pair (X,p) is called a partial metric space in short PMS and p is called a partial metric on X.

Let $X = \{2,3\}$ be a non empty set and let $p : X \times X \to \mathbb{R}_0^+$ such that $p(x,y) = \min\{|x-y|, |x+y|\}$, then we see that (X,p) is a partial metric space.

Let (X,p) be a PMS. Then the functions $d_p, d_m : X \times X \to \mathbb{R}_0^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - P(y,y)$$

and

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\},$$

are usual metrics on X. It is clear that d_p and d_m are equivalent metric. It can be easily deduced that $d_p(x,y) = |x-y| = d_m(x,y)$.

We are giving some definitions were defined by Matthews {[8]&[9]} as follows.

Definition 1.2. *{[4],[8] & [9]} (i) A sequence $\{x_n\}$ in a PMS (X,d) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.*

(ii) A sequence $\{x_n\}$ in a PMS (X,d) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite).

(iii) A PMS (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0,\varepsilon))$.

We introduce following lemmas which we will use in proof of the main results.

Lemma 1.1. *{[4],[8]& [9]} (A) A sequence $\{x_n\}$ is Cauchy in a PMS (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space $\left(X,d_p\right)$.*

(B) A PMS (X,p) is complete if and only if the metric space $\left(X,d_p\right)$ is complete. Moreover, $\lim_{n\to\infty} p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n\to\infty} p(x_n,x)$, where x is a limit of the sequence $\{x_n\}$ in $\left(X,d_p\right)$.

Remark 1.1. *Let (X,p) be a PMS. Therefore, we have*
(A) if $p(x,y) = 0$, then $x = y$ and
(B) if $x \ne y$, then $p(x,y) > 0$.

Lemma 1.2. *[5] Assume $x_n \to z$ as $n \to \infty$ in a PMS (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

Karapinar and Yuksel[6] investigated some common fixed point theorems in partial metric spaces and obtained the following theorem.

Theorem 1.1. *[6] Suppose that (X,p) is a complete PMS and T,S are self-mappings on X. If there exists an $r \in (0,1)$ such that*

$$p(Tx,Sy) \le rM(x,y),$$

for any $x,y \in X$, where

$$M(x,y) = \max\left\{p(Tx,x), p(Sy,y), p(x,y), \frac{1}{2}\left[p(Tx,y) + p(Sy,x)\right]\right\},$$

then there exists $z \in X$ such that $Tz = Sz = z$.

Jebril et al.[3] have investigated some common fixed point theorems under rational contraction for a pair of mappings in bicomplex valued metric spaces and obtained the following result.

Theorem 1.2. *[3] Let (X,d) be a complete bicomplex valued metric space and the mappings S,T : $\mathbb{C}_2 \to \mathbb{C}_2$ satisfy*

$$d(Sz;Tz') \leq \alpha d(z,z') + \frac{\beta d(z,Sz)\, d(z',Tz')}{d(z,Tz') + d(z'Sz) + d(z,z')},$$

for all $z,z' \in X$ such that $d(z,Tz') + d(z'Sz) + d(z,z') \neq 0$ with $z \neq z'$, where α,β are non negative real numbers with $\alpha + \beta < 1$ or $d(z,z') = 0$ if $d(z,Tz') + d(z'Sz) + d(z,z') = 0$. Then S and T have a unique common fixed point.

Our results are the extension of Theorem 1.1 and modification of Theorem 1.2. Also we have taken some concepts from the results of Jebril et al. [4].

2 Main Results

In this section we have proved two theorems, the first theorem as an extension of Theorem 1.1 and the second theorem as a modification of Theorem 1.2.

Theorem 2.1. *Let (X,p) be a partial metric space and the mappings $S,T : X \to X$ satisfy*

$$p(Tx,Sy)$$

$$\leq h \max\left\{\frac{p(x,y)+p(x,Tx)}{2}, \frac{p(x,y)+p(y,Sy)}{2}, p(x,Tx)+p(y,Sy)-p(x,y)\right\}, \tag{2.1}$$

for all $x,y \in X$ where $0 < h < 1$. Then S and T have a common fixed point in X.

Proof. We choose an arbitrary point $x_0 \in X$. Sequence $\{x_n\}$ can be formed in X such that $Sx_0 = x_1,\ Tx_1 = x_2,\ Sx_2 = x_3,\ Tx_3 = x_4, ...$

$$\text{i.e.,}\quad Sx_{2n} = x_{2n+1}, Tx_{2n+1} = x_{2n+2} \quad \text{for} \quad n = 0,1,2,... \tag{2.2}$$

If there exists a positive integer n_0 such that $x_{2n_0} = x_{2n_0+1}$, then x_{2n_0} is a fixed point of S and hence a fixed point of T. Indeed, since $x_{2n_0} = x_{2n_0+1} = Sx_{2n_0}$, then

$$x_{2n_0+1} = x_{2n_0} = Sx_{2n_0} = Sx_{2n_0+1}. \tag{2.3}$$

We have to show that the sequence $\{x_n\}$ is Cauchy sequence. For putting $x = x_{2k}$ and $y = x_{2k+1}$ in (2.1), we have

$$p(x_{2n_0+2}, x_{2n_0+1}) = p(Tx_{2n_0+1}, Sx_{2n_0})$$

$$\begin{aligned}
&\leq h \max\left\{\begin{array}{c} \frac{1}{2}\left(p(x_{2n_0+1},x_{2n_0}) + p(x_{2n_0+1},Tx_{2n_0+1})\right), \\ \frac{1}{2}\left(p(x_{2n_0+1},x_{2n_0}) + p(x_{2n_0},Sx_{2n_0})\right), \\ \left(p(x_{2n_0+1},Tx_{2n_0+1}) + p(x_{2n_0},Sx_{2n_0}) - p(x_{2n_0+1},x_{2n_0})\right) \end{array}\right\} \\
&= h \max\left\{\begin{array}{c} \frac{1}{2}\left(p(x_{2n_0+1},x_{2n_0}) + p(x_{2n_0+1},x_{2n_0+2})\right), \\ \frac{1}{2}\left(p(x_{2n_0+1},x_{2n_0}) + p(x_{2n_0},x_{2n_0+1})\right), \\ \left(p(x_{2n_0+1},x_{2n_0+2}) + p(x_{2n_0},x_{2n_0+1}) - p(x_{2n_0+1},x_{2n_0})\right) \end{array}\right\} \\
&= h \max\left\{\begin{array}{c} \frac{1}{2}\left(p(x_{2n_0+1},x_{2n_0}) + p(x_{2n_0+1},x_{2n_0+2})\right), \\ p(x_{2n_0},x_{2n_0+1}), p(x_{2n_0+1},x_{2n_0+2}) \end{array}\right\} \\
&\leq h \max\left\{p(x_{2n_0},x_{2n_0+1}), p(x_{2n_0+1},x_{2n_0+2})\right\}, \text{ as } \frac{1}{2}(a+b) \leq \max(a,b) \\
&= h \max\left\{p(x_{2n_0},x_{2n_0}), p(x_{2n_0+1},x_{2n_0+2})\right\}, \text{ as } x_{2n_0} = x_{2n_0+1} \text{ by assumption} \\
&= h \cdot p(x_{2n_0+1},x_{2n_0+2}), \text{ by } (PM2).
\end{aligned}$$

Therefore we have $(1-h)\cdot p(x_{2n_0+1}, x_{2n_0+2}) \leq 0$, then $p(x_{2n_0+1}, x_{2n_0+2}) = 0$, since $h < 1$, which yields that $Tx_{2n_0+1} = x_{2n_0+2} = x_{2n_0+1}$ i.e., x_{2n_0+1} is a fixed point of T. Also x_{2n_0} is a fixed point of S. Therefore x_{2n_0} $(= x_{2n_0+1})$ is a common fixed point of S and T. A similar conclusion holds if it is seen that $x_{2n_0+1} = x_{2n_0+2}$ for some positive integer n_0. Therefore we may assume that $x_n \neq x_{n+1}$ for all n.

Case-I:

Let n be an odd number. Then by (2.1) we have

$$\begin{aligned}
&p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1})\\
&\leq h\max\left\{\begin{array}{c}\frac{1}{2}(p(x_n, x_{n+1}) + p(x_n, Tx_n)), \frac{1}{2}(p(x_n, x_{n+1}) + p(x_{n+1}, Sx_{n+1})),\\ (p(x_n, Tx_n) + p(x_{n+1}, Sx_{n+1}) - p(x_n, x_{n+1}))\end{array}\right\}\\
&= h\max\left\{\begin{array}{c}\frac{1}{2}(p(x_n, x_{n+1}) + p(x_n, x_{n+1})), \frac{1}{2}(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})),\\ (p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_n, x_{n+1}))\end{array}\right\}\\
&= h\max\left\{p(x_n, x_{n+1}), \frac{1}{2}(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})), p(x_{n+1}, x_{n+2})\right\}\\
&\leq h\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}\ \text{, as} \frac{1}{2}(a+b) \leq \max\{a, b\}\\
&= h\cdot p(x_n, x_{n+1}).
\end{aligned}$$

Because, if we take $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2})$, then above inequality yields $x_{n+1} = x_{n+2}$, which contradicts our assumption that $x_n \neq x_{n+1}$ for all n.

Case-II

If n is even number, then by (2.1) we get that

$$p(x_{n+2}, x_{n+1}) = p(Tx_{n+1}, Sx_n)$$

$$\begin{aligned}
&\leq h\max\left\{\begin{array}{c}\frac{1}{2}(p(x_{n+1}, x_n) + p(x_{n+1}, Tx_{n+1})), \frac{1}{2}(p(x_{n+1}, x_n) + p(x_n, Sx_n)),\\ (p(x_{n+1}, Tx_{n+1}) + p(x_n, Sx_n) - p(x_{n+1}, x_n))\end{array}\right\}\\
&= h\max\left\{\begin{array}{c}\frac{1}{2}(p(x_{n+1}, x_n) + p(x_{n+1}, x_{n+2})), \frac{1}{2}(p(x_{n+1}, x_n) + p(x_n, x_{n+1})),\\ (p(x_{n+1}, x_{n+2}) + p(x_n, x_{n+1}) - p(x_n, x_{n+1}))\end{array}\right\}\\
&= h\max\left\{\frac{1}{2}(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\right\}\\
&\leq h\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}\ \text{, as } \frac{1}{2}(a+b) \leq \max\{a, b\}\\
&= h\cdot p(x_n, x_{n+1}).
\end{aligned}$$

Here also $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_n, x_{n+1})$ is taken for the similar reason as odd number. Therefore the inequality

$$p(x_{n+1}, x_{n+2}) \leq hp(x_n, x_{n+1}), \tag{2.4}$$

holds for all $n = 0, 1, 2, ...$ Hence $\{p(x_n, x_{n+1})\}$ is a non increasing sequence of non negative real numbers. Using the result(2.4), it can easily be proved that

$$p(x_n, x_{n+1}) \leq h^n p(x_0, x_1),\ \forall\, n = 0, 1, 2, ... \tag{2.5}$$

Now let us consider that

$$d_p(x_{n+1}, x_{n+2}) = 2p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) \leq 2p(x_{n+1}, x_{n+2}),$$

$$\text{i.e., } d_p(x_{n+1}, x_{n+2}) \leq 2h^{n+1}1p(x_0, x_1). \tag{2.6}$$

Since $0 < h < 1$, from (2.5) and (2.6) we get that $\lim_{n\to\infty} d_p(x_{n+1}, x_{n+2}) = 0$. Moreover by 2.6 we have

$$\begin{aligned} d_p(x_{n+1}, x_{n+s}) &\leq d_p(x_{n+1}, x_{n+2}) + ... + d_p(x_{n+s-1}, x_{n+s}) \\ &\leq 2h^{n+1} p(x_0, x_1) + ... + 2h^{k+s} p(x_0, x_1). \end{aligned}$$

After standard calculation, we obtain that $\{x_n\}$ is a Cauchy sequence in (X, d_p) i.e., $d_p(x_n, x_m) \to 0$ as $n, m \to \infty$. Since (X, p) is complete, by Lemma 1.1 we have (X, d_p) is complete and the sequence $\{x_n\}$ is convergent in (X, d_p) to $z \in X$ (say)

Again by Lemma 1.1, we have

$$p(z,z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.7}$$

Since $\{x_n\}$ is a Cauchy sequence in (X, d_p), therefore we have $\lim_{n,m\to\infty} d_p(x_n, x_m) = 0$. We assert that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Without loss of generality, let us assume that $m > n$. Then we observe that

$$p(x_n, x_{n+2}) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}),$$

$$\text{i.e., } p(x_n, x_{n+2}) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}). \tag{2.8}$$

Using 2.8 it follows that

$$\begin{aligned} p(x_n, x_{n+3}) &\leq p(x_n, x_{n+2}) + p(x_{n+2}, x_{n+3}) - p(x_{n+2}, x_{n+2}) \\ &\leq p(x_n, x_{n+2}) + p(x_{n+2}, x_{n+3}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}). \end{aligned}$$

Therefore by induction we obtain that

$$p(x_n, x_m) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + ... + p(x_{m-1}, x_m). \tag{2.9}$$

With the help of (2.4), the expression (2.9) reduces to

$$p(x_n, x_m) \leq h^n p(x_0, x_1) + h^{n+1} p(x_0, x_1) + ... + h^{m-1} p(x_0, x_1),$$

$$\text{i.e., } p(x_n, x_m) \leq h^n(1 + h + ... + h^{m-n-1}) p(x_0, x_1). \tag{2.10}$$

(2.11) Since $h < 1$, taking $n, m \to \infty$ we can make $h^n \to 0$ and so $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Therefore, from (2.7), we have

$$p(z,z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.12}$$

We assert that $Tz = z$. On the contrary, assume $Tz \neq z$. Then $p(z, Tz) > 0$. Let $\{x_{2n(i)}\}$ be a subsequence of $\{x_{2n}\}$ and hence of $\{x_n\}$. Due to (2.1) and (2.2) we have

$$p(x_{2n(i)+1}, Tz) = p(Sx_{2n(i)}, Tz) = p(Tz, Sx_{2n(i)})$$

$$\leq h \max \left\{ \begin{array}{c} \frac{1}{2}(p(z, x_{2n(i)}) + p(z, Tz)), \frac{1}{2}(p(z, x_{2n(i)}) + p(x_{2n(i)}, Sx_{2n(i)})), \\ (p(z, Tz) + p(x_{2n(i)}, Sx_{2n(i)}) - p(z, x_{2n(i)})) \end{array} \right\}.$$

Taking limit as $n \to \infty$, we get

$$p(z, Tz) \leq h \max\left\{\frac{1}{2}(0 + p(z, Tz)), \frac{1}{2}(0 + 0), (p(z, Tz) + 0 - 0)\right\} = hp(z, Tz).$$

Thus $p(z, Tz) \leq hp(z, Tz)$. Since $h < 1$, therefore $p(z, Tz) = 0$ and hence by Remark(2.1), we get $Tz = z$. Analogously, if we choose a subsequence $\{x_{2n(i)+1}\}$ of $\{x_{2n+1}\}$, we can prove that $Sz = z$. Hence z is a common fixed point of S and T i.e., $Tz = Sz = z$.

This completes the proof of the theorem. □

Theorem 2.2. *Let (X, p) be a complete partial metric space and let the mappings $S, T : X \to X$ satisfy*

$$p(Sx, Ty) \leq \alpha p(x, y) + \beta \frac{p(x, y)\, p(x, Ty) + p(x, y)\, p(y, Sx)}{p(x, Sx) + p(y, Ty)}, \tag{2.13}$$

for all $x, y \in X$ such that $x \neq y$, where α and β are non negative real numbers with $\alpha + \beta < \frac{1}{2}$. If $p(x, Ty)$, $p(y, Sx)$, $p(x, Sx)$ and $p(y, Ty)$ satisfy the condition

$$p(x, Ty) + p(y, Sx) \leq 2\{p(x, Sx) + p(y, Ty)\}, \tag{2.14}$$

then the pair (S, T) have a unique common fixed point in X.

Proof. Let us choose an arbitrary point $x_0 \in X$. Sequence $\{x_n\}$ can be formed in X such that $Sx_0 = x_1,\ Tx_1 = x_2,\ Sx_2 = x_3,\ Tx_3 = x_4, \ldots$i.e.,

$$Sx_{2n} = x_{2n+1}, Tx_{2n+1} = x_{2n+2}.$$

Then from (2.13)we get

$$p(x_{2n+1}, x_{2n+2}) = p(Sx_{2n}, Tx_{2n+1})$$
$$\leq \alpha p(x_{2n}, x_{2n+1}) + \beta \frac{p(x_{2n}, x_{2n+1})\, p(x_{2n}, Tx_{2n+1}) + p(x_{2n}, x_{2n+1})\, p(x_{2n+1}, Sx_{2n})}{p(x_{2n}, Sx_{2n}) + p(x_{2n+1}, Tx_{2n+1})},$$

$$\text{i.e., } p(x_{2n+1}, x_{2n+2}) \leq \alpha p(x_{2n}, x_{2n+1}) + \beta p(x_{2n}, x_{2n+1}) \frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}. \tag{2.15}$$

Also from triangle property of PMS we get that

$$p(x_{2n}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) - p(x_{2n+1}, x_{2n+1}),$$

which implies that

$$\frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})} \leq 1. \tag{2.16}$$

Hence from (2.13) and (2.16), we obtain that

$$p(x_{2n+1}, x_{2n+2}) \leq (\alpha + \beta)\, p(x_{2n}, x_{2n+1}) = hp(x_{2n}, x_{2n+1}),$$

where $h = \alpha + \beta < \frac{1}{2}$,

$$\text{i.e., } p(x_{2n+1}, x_{2n+2}) \leq hp(x_{2n}, x_{2n+1}). \tag{2.17}$$

Again we have that

$$p(x_{2n+3}, x_{2n+2}) = p(Sx_{2n+2}, Tx_{2n+1}) \leq \alpha p(x_{2n+2}, x_{2n+1})$$
$$+\beta \frac{p(x_{2n+2}, x_{2n+1})\, p(x_{2n+2}, Tx_{2n+1}) + p(x_{2n+2}, x_{2n+1})\, p(x_{2n+1}, Sx_{2n})}{p(x_{2n+2}, Sx_{2n+2}) + p(x_{2n+1}, Tx_{2n+1})},$$

$$\leq \alpha p(x_{2n+2}, x_{2n+1}) + \beta p(x_{2n+2}, x_{2n+1}) \frac{p(x_{2n+2}, x_{2n+2}) + p(x_{2n+1}, x_{2n+3})}{p(x_{2n+2}, x_{2n+3}) + p(x_{2n+1}, x_{2n+2})}. \tag{2.18}$$

Using the condition $(PM4)$, we get that

$$\frac{p(x_{2n+2}, x_{2n+2}) + p(x_{2n+1}, x_{2n+3})}{p(x_{2n+2}, x_{2n+3}) + p(x_{2n+1}, x_{2n+2})} \leq 1. \tag{2.19}$$

Therefore by 2.19 we get that

$$\begin{aligned} p(x_{2n+3}, x_{2n+2}) &\leq \alpha p(x_{2n+2}, x_{2n+1}) + \beta p(x_{2n+2}, x_{2n+1}) \\ &= (\alpha + \beta)\, p(x_{2n+2}, x_{2n+1}), \end{aligned}$$

Therefore the inequality $p(x_{n+1}, x_{n+2}) \leq hp(x_n, x_{n+1})$ holds for all $n = 0, 1, 2, ...$ Hence $\{p(x_n, x_{n+1})\}$ is a non increasing sequence of non negative real numbers. Using (2.6), (2.7) and (2.10), we get

$$p(x_n, x_m) \leq h^n \frac{1}{1-h} p(x_0, x_1).$$

Again using (2.12), we obtain that

$$p(u, u) = \lim_{n,m\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Now we assert that $p(u, Su) = 0$. If possible, let $p(u, Su) > 0$. Then

$$\begin{aligned} p(u, Su) &\leq p(u, x_{2n+2}) + p(x_{2n+2}, Su) - p(x_{2n+2}, x_{2n+2}) \\ &= p(u, x_{2n+2}) + p(Tx_{2n+1}, Su) - p(x_{2n+2}, x_{2n+2}) \\ &\leq p(u, x_{2n+2}) + \alpha p(x_{2n+1}, u) - p(x_{2n+2}, x_{2n+2}) \\ &\quad +\beta \frac{p(x_{2n+1}, u)\, p(x_{2n+1}, Tu) + p(x_{2n+1}, u)\, p(u, Sx_{2n+1})}{p(x_{2n+1}, Sx_{2n+1}) + p(u, Tu)} \\ &\leq p(u, x_{2n+2}) + \alpha p(x_{2n+1}, u) - p(x_{2n+2}, x_{2n+2}) \\ &\quad +\beta p(x_{2n+1}, u) \frac{p(x_{2n+1}, Tu) + p(u, x_{2n+2})}{p(x_{2n+1}, x_{2n+2}) + p(u, Tu)}. \end{aligned}$$

Taking limit as $n \to \infty$, we get that

$$\begin{aligned} p(u, Su) &\leq p(u, u) - p(u, u) + \alpha p(u, u) + \beta p(u, u) \frac{p(u, Tu) + p(u, u)}{p(u, u) + p(u, Tu)} \\ &= (\alpha + \beta)\, p(u, u) < p(u, u), \text{ as } (\alpha + \beta) < \frac{1}{2} \end{aligned}$$

which contradicts the condition $(PM2)$. Therefore $p(u, Su) = 0$. Hence u is a fixed point of S. Similarly, it can be shown that u is also a fixed point of T . Hence u is a common fixed point of S and T.

Uniqueness:

If possible let u^* be another common fixed point of T and S. Now we see that

$$\begin{aligned} p(u,u^*) &\le p(u,x_{2n+1}) + p(x_{2n+1},u^*) - p(x_{2n+1},x_{2n+1}) \\ &= p(u,x_{2n+1}) + p(Su,Tu^*) - p(x_{2n+1},x_{2n+1}) \\ &\le p(u,x_{2n+1}) + \alpha p(u,u^*) + \beta \frac{p(u,u^*)\,p(u,Tu^*) + p(u,u^*)\,p(u^*,Su)}{p(u,Su)+p(u^*,Tu^*)} \\ &- p(x_{2n+1},x_{2n+1}) \\ &= p(u,x_{2n+1}) + \alpha p(u,u^*) + \beta p(u,u^*)\,\frac{p(u,Tu^*)+p(u^*,Su)}{p(u,Su)+p(u^*,Tu^*)} \\ &- p(x_{2n+1},x_{2n+1}). \end{aligned}$$

Taking limit as $n \to \infty$, we get that

$$p(u,u^*) \le p(u,u) + \alpha p(u,u^*) + \beta p(u,u^*)\,\frac{p(u,Tu^*)+p(u^*,Su)}{p(u,Su)+p(u^*,Tu^*)} - p(u,u). \tag{2.20}$$

Also from (2.14), we get that

$$p(u,Tu^*) + p(u^*,Su) \le 2\{p(u,Su)+p(u^*,Tu^*)\},$$

which implies that

$$\frac{p(u,Tu^*)+p(u^*,Su)}{p(u,Su)+p(u^*,Tu^*)} \le 2. \tag{2.21}$$

Using condition (2.21) in (2.20), we get that

$$\begin{aligned} p(u,u^*) &\le \alpha p(u,u^*) + 2\cdot\beta p(u,u^*) \le 2\cdot(\alpha+\beta)\,p(u,u^*) \\ &\Rightarrow (1-2\cdot(\alpha+\beta))\,p(u,u^*) \le 0 \\ &\Rightarrow p(u,u^*) \le 0 \text{ as } (\alpha+\beta) < \frac{1}{2}. \end{aligned}$$

Thus we conclude that $p(u,u^*) = 0$, which shows that $u = u^*$. Hence u is a unique common fixed point of S and T. Hence the proof of the theorem is established. □

3 Future Prospect

In the line of works as carried out in the paper one may think of the deduction of fixed point theorems under different conditions in complex valued partial metric spaces and this may be an active area of research to the future workers in this branch.

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