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COMMON FIXED POINT THEOREMS FOR CONVERSE COMMUTING MAPPINGS IN BICOMPLEX VALUED METRIC SPACES

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Abstract: The main purpose of this paper is to prove some common fixed point theorems for converse commuting self-maps for non-complete bicomplex valued metric spaces. Our results are the generalisations of the results of Chauhan & Sahper (2013) and Kumar et al. (2014). Moreover, some concepts of Choi et al., proved some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces published in Honam Mathematical Journal in 2017 and Jebril et al., proved common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces published in Journal of Interdisciplinary Mathematics in 2019 are used here.

Keywords and Phrases: Bicomplex valued metric space, converse commuting mappings, common fixed point.

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1. Introduction, Preliminaries and Definitions

The Banach contraction principle [5] is one of the most important and useful results in fixed point theory. Although the fixed point theory was first introduced by Brouwer [4] in 1912, but the credit of making the concept useful and popular goes to Polish mathematician Banach [5] in 1922. The famous Banach contraction mapping theorem states that

"Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X, that is there is a constant $\lambda \in [0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$.Then T has a unique fixed point in X."

Ajam et al [1] introduced a new space called complex-valued metric space which is more general than the well known metric space and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings involving rational expression. Kakutani [12] and Kannan [13] respectively generalised the results obtained by of Brouwer and Banach. Complex and bicomplex valued metric spaces are in fact deduced from cone metric spaces $\{cf.$ [2], [3], [8], [11], [15], [16], [20], [22], [24]$\}$.

Lii [17] introduced the concept of coverse commuting maps and proved fixed point theorems for single valued maps in metric spaces. Lii & Hu [18] and Popa [19] introduced converse commuting multivalued mappings and proved some fixed point theorems for converse commuting multivalued mappings. Kumar et al.[14] proved some common fixed point theorems for converse commuting mappings in complex valued metric spaces.

We write the set of real, complex and bicomplex number respectively as $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$.

The partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, (ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$, (iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$, (iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (ii), (iii) and (iv) is satisfied and we write $z_1 \prec z_2$ if only (iv) is satisfied. Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}_1$ is defined as follows:

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$,

(2) If $z_1 \precsim z_2, z_2 \precsim z_3$ then $z_1 \precsim z_3$ and

(3) If $z_1 \precsim z_2$ and $\lambda < 1$ is a non-negative real number then $\lambda z_1 \precsim z_2$.

Azam et. al. [1] defined the complex valued metric space in the following way:

Definition 1.1. [1] *Let X be a non empty set and the mapping $d : X \times X \to \mathbb{C}_1$, satisfies the following conditions:*

(d_1) $0 \precsim d(x,y)$, *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,
(d_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *and*
(d_3) $d(x,y) \precsim d(x,z) + d(z,y)$, *for all* $x, y, z \in X$.

Then d *is called a complex valued metric on* X *and* (X,d) *is called a complex valued metric space.*

The space $\mathbb{C}_2$ is the first in an infinite sequence of multicomplex spaces which are generalizations of $\mathbb{C}_1$.

The notion of the space $\mathbb{C}_2$ was defined by Segre [23] as

$$\mathbb{C}_2 = \{w : w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3, p_k \in \mathbb{C}_0,\ 0 \leq k \leq 3\}$$

$$\text{i.e., } \mathbb{C}_2 = \{w = z_1 + i_2 z_2 \mid z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = p_0 + i_1 p_1,\ z_2 = p_2 + i_1 p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product of all units are commutative and satisfy

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

Definition 1.2. *For a bicomplex number* $w = z_1 + i_2 z_2$, *the norm is denoted by* $\|z_1 + i_2 z_2\|$ *and defined by*

$$\|z_1 + i_2 z_2\| = \left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}} = \left(|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2\right)^{\frac{1}{2}}.$$

If we take $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$ *for* $p_k \in \mathbb{C}_0,\ k = 0, 1, 2, 3$ *then the norm of* w *is defined by*

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}}.$$

The partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ was defined by Choi et. al. [7] as $u \precsim_{i_2} v$ if and only if $u_1 \precsim u_2$ and $v_1 \precsim v_2$, where $u_1, u_2, v_1, v_2 \in \mathbb{C}$. The bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ on a non-empty set X and the structure (X,d) on $\mathbb{C}_2$ were defined by Choi et. al. [7] accordingly.

By the deduction of Rochon & Shapiro [21] we get the results
(i) $\|uv\| \leq \sqrt{2}\,\|u\|\,\|v\|$ for any u, $v \in \mathbb{C}_2$; .
(ii) $\|uv\| = \|u\|\,\|v\|$ for any u, $v \in \mathbb{C}_2$ with at least one of them is degenerated;
(iii) $\left\|\frac{1}{u}\right\| = \frac{1}{\|u\|}$ for any degenerated bicomplex number u with $0 \precnsim_{i_2} u$.

Definition 1.3. [17] *A point* $x \in X$ *is said to be commuting point of* $f, g : X \to X$ *if* $fgx = gfx$.

Definition 1.4. [17] *Two maps* $f, g : X \to X$ *are said to be converse commuting* *if $fgx = gfx$ implies $fx = gx$.The set of converse commuting points of f and g is denoted by $C(f,g)$.*

Choi et al. [7] defined the 'max" function on $\mathbb{C}_2$ as follows:

Definition 1.5. *The max function for the partial order $\precsim_{i_2}$ on $\mathbb{C}_2$ is defined as follows:*

(i) $\max\{u,v\} = v, u \precsim_{i_2} v$;

(ii) $u \precsim_{i_2} \max\{u,v\}$ *implies* $u \precsim_{i_2} v$ *or* $\|u\| \leq \|v\|$;

(iii) $\max\{u,v\} = v$ *iff* $u \precsim_{i_2} v$ *or* $\|u\| \leq \|v\|$.

In this paper we prove some common fixed point theorems using converse commuting mappings on bicomplex-valued metric spaces. Our results are the generalisations of the results of Chauhan & Sahper [6] and Kumar et.al. [14] using the concepts from [9], [10].

2. Main Results

In this section we prove some common fixed point theorems using converse commuting mappings.

Theorem 2.1. *Let (X,d) be a bicomplex valued metric space and A,B,S and $T : X \to X$ be mappings such that the pair (A,S) and (B,T) be conversely commuting and satisfying the condition*

$$d(Ax,By) \precsim_{i_2} \lambda \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty)\} \tag{2.1}$$

for all $x,y \in X\,(\subseteq \mathbb{C}_2)$, $0 \leq \lambda < 1$. If A and S have a commuting point and B and T have a commuting point, then A,B,S and T have a unique common fixed point in X.

Proof. Let u be the commuting point of A and S, then $ASu = SAu$. Again let v be the commuting point of B and T, then $BTv = TBv$. Since A and S are conversely commuting, then we have $Au = Su$. Again since B and T are conversely commuting, then we have $Bv = Tv$. Hence $AAu = ASu = SAu = SSu$ and $BBv = BTv = TBv = TTv$.

We claim that $Au = Bv$. If not then putting $x = u$, $y = v$ in (2.1) we get that

$$\begin{aligned} d(Au,Bv) &\precsim_{i_2} \lambda \max\{d(Su,Tv),\ d(Au,Su),\ d(Bv,Tv)\} \\ &\precsim_{i_2} \lambda \max\{d(Su,Tv),\ 0,\ 0\} \end{aligned}$$

$$i.e.,\ \|d(Au,Bv)\| \leq \lambda \|d(Au,Bv)\|,$$

implying that $\|d(Au,Bv)\| = 0$, as $\lambda < 1$. Hence $Au = Bv$ *i.e.,* $Au = Su = Bv = Tv$.

Now we show that Au is a fixed point of the mapping A. Putting $x = Au$ and $y = v$ in (2.1) we obtain that

$$\begin{aligned}
d(AAu, Bv) &\precsim_{i_2} \lambda \max\{d(SAu, Tv), d(AAu, SAu), d(Bv, Tv)\} \\
i.e., d(AAu, Au) &\precsim_{i_2} \lambda \max\{d(ASu, Tv), d(AAu, ASu), d(Bv, Bv)\} \\
i.e.,\ d(AAu, Au) &\precsim_{i_2} \lambda \max\{d(AAu, Au), d(AAu, AAu), d(Bv, Bv)\} \\
i.e.,\ d(AAu, Au) &\precsim_{i_2} \lambda \max\{d(AAu, Au), 0, 0\},
\end{aligned}$$

which implies that $\|d(AAu, Au)\| = 0$, as $\lambda < 1$.

Again putting $x = u$ and $y = Bv$ in (2.1) we get that

$$\begin{aligned}
d(Au, BBv) &\precsim_{i_2} \lambda \max\{d(Su, TBv),\ d(Au, Su),\ d(BBv, Tbv)\} \\
i.e.,\ d(Au, BBv) &\precsim_{i_2} \lambda \max\{d(Bv, BBv),\ d(Au, Su),\ d(BBv, BBv)\} \\
i.e.,\ d(Bv, BBv) &\precsim_{i_2} \lambda \max\{d(Bv, BBv),\ 0,\ 0\}.
\end{aligned}$$

Therefore,

$$\|\ d(Bv, BBv)\| \precsim_{i_2} \lambda \|\ d(Bv, BBv)\|,$$

which is a contradiction. Therefore $\|\ d(Bv, BBv)\| = 0$, *i.e.*, $BBv = Bv$.

Thus we have $Au = Bv = BBv = BAu$ and hence Au is a fixed point of the mapping B. On the other hand, $Au = AAu = ASu = SAu$. Hence $Au = w$ is a common fixed point of A, B, S and T in X.

Uniqueness:

If possible suppose that w^* be another common fixed point of the mappings A, B, S and T.

Then using (2.1) we get that

$$\begin{aligned}
d(Aw, Bw^*) &\precsim_{i_2} \lambda \max\{d(Sw, Tw^*), d(Aw, Sw), d(Bw^*, Tw^*)\} \\
i.e., d(w, w^*) &\precsim_{i_2} \lambda d(w, w^*),
\end{aligned}$$

which implies that $\|d(w, w^*)\| \leq \lambda \|d(w, w^*)\|$, *i.e.*, $\|d(w, w^*)\| = 0$ implies $w = w^*$.

Therefore w is a unique common fixed point of A, B, S and T.

This completes the proof of the theorem.

Corollary 2.1. *Let (X, d) be a bicomplex valued metric space and A and $S : X \to X$ be mappings such that the pair (A, S) be conversely commuting and satisfying the condition*

$$d(Ax, Ay) \precsim_{i_2} \lambda \max\{d(Sx, Sy),\ d(Ax, Sx),\ d(Ay, Sy)\}$$

for all $x, y \in X$ *and* $0 \leq \lambda < 1$. *If* A *and* S *have a commuting point then* A, B, S *and* T *have a unique common fixed point in* X *where* $X \subseteq \mathbb{C}_2$.

Proof. The proof can be established easily by taking $A = B$ and $S = T$ in the proof of Theorem 2.1.

Theorem 2.2. *Let* (X, d) *be a bicomplex valued metric space and let* f, g, h *and* k *be self-mappings defined on* X *such that the pairs* (f, k) *and* (g, h) *be conversely commuting and satisfying*

$$d(fz, gz') \precsim_{i_2} \lambda \max\left\{ \frac{d(kz,hz')+d(kz,fz)}{2}, \frac{d(kz,fz)+d(hz',gz')}{2}, \frac{d(kz,gz')+d(hz',fz)}{2} \right\} \tag{2.2}$$

for all $z, z' \in X \, (\subseteq \mathbb{C}_2)$, $\lambda \in (0, 1)$. *If the pairs* (f, k) *and* (g, h) *have a commuting point then* f, g, h *and* k *have a unique common fixed point in* X.

Proof. Let $u \in C(f, k)$ and $v \in C(g, h)$, therefore $fku = kfu$ implies $fu = ku$. Similarly $ghv = hgv$ implies $gv = hv$.

First we prove that $fu = gv$. If not then using (2.2) for $z = u, z' = v$ we get that

$$\begin{aligned} d(fu, gv) &\precsim_{i_2} \lambda \max\left\{ \frac{d(ku,hv)+d(ku,fu)}{2}, \frac{d(ku,fu)+d(hv,gv)}{2}, \frac{d(ku,gv)+d(hv,fu)}{2} \right\} \\ &\precsim_{i_2} \lambda \max\left\{ \frac{d(fu,gv)+d(fu,fu)}{2}, \frac{d(fu,fu)+d(gv,gv)}{2}, \frac{d(fu,gv)+d(gv,fu)}{2} \right\} \\ &\precsim_{i_2} \lambda \max\left\{ \frac{d(fu, gv)}{2}, 0, d(fu, gv) \right\}, \end{aligned}$$

which implies that

$$d(fu, gv) \precsim \lambda d(fu, gv)$$

i.e.

$$\|d(fu, gv)\| \leq \lambda \|d(fu, gv)\|,$$

which is a contradiction. Therefore $\|d(fu, gv)\| = 0$, *i.e.*, $fu = gv$.

Now we claim that $f^2u = fu$. If not then using (2.2) for $z = fu, z' = v$ we have

$$\begin{aligned} d(f^2u, gv) &\precsim_{i_2} \lambda \max\left\{ \frac{d(kfu,hv)+d(kfu,ffu)}{2}, \frac{d(kfu,ffu)+d(hv,gv)}{2}, \frac{d(kfu,gv)+d(hv,ffu)}{2} \right\} \\ &\precsim_{i_2} \lambda \max\left\{ \frac{d(fku,gv)+d(fku,ffu)}{2}, \frac{d(fku,ffu)+d(gv,gv)}{2}, \frac{d(fku,gv)+d(gv,ffu)}{2} \right\} \end{aligned}$$

$$\begin{aligned}
&\precsim_{i_2} \lambda \max\left\{ \frac{d(ffu,gv)+d(ffu,ffu)}{2}, \frac{d(ffu,ffu)+d(gv,gv)}{2}, \frac{d(ffu,gv)+d(gv,ffu)}{2} \right\} \\
&\precsim_{i_2} \lambda \max\left\{ \frac{d(ffu,gv)}{2}, 0, d(ffu,gv) \right\} \\
&\precsim_{i_2} \lambda \max\left\{ \frac{d(f^2u,fu)}{2}, 0, d(f^2u,fu) \right\},
\end{aligned}$$

which implies that

$$d(f^2u, fu) \precsim \lambda d(f^2u, fu)$$

i.e.

$$\left\|d(f^2u, fu)\right\| \precsim \lambda \left\|d(f^2u, fu)\right\|,$$

which is a contradiction. Therefore $\|d(f^2u, fu)\|$, i.e., $f^2u = fu$. Similarly we get that $g^2v = gv$, $fu = ffu = fku = kfu$, $gv = ggv = ghv = hgv$.

Thus f, g, h and k have a common fixed point in X. Let $w = fu$ is a common fixed point of f, g, h and k in X.

Now we show that w is a unique common fixed point. If possible suppose that w' be another common fixed point in X. Then using (2.2) we get that

$$\begin{aligned}
d(w, w') &= d(fw, gw') \\
&\precsim_{i_2} \lambda \max\left\{ \frac{d(kw,hw')+d(kw,fw)}{2}, \frac{d(kw,fw)+d(hw',gw')}{2}, \frac{d(kw,gw')+d(hw',fw)}{2} \right\} \\
&\precsim_{i_2} \lambda \max\left\{ \frac{d(w,w')+d(w,w)}{2}, \frac{d(w,w)+d(w',w')}{2}, \frac{d(w,w')+d(w',w)}{2} \right\} \\
&\precsim_{i_2} \lambda \max\left\{ \frac{d(w,w')}{2}, 0, d(w,w') \right\}
\end{aligned}$$

which implies that

$$d(w, w') \precsim_{i_2} \lambda d(w, w')$$

i.e.,

$$\|d(w, w')\| \leq \lambda \|d(w, w')\|,$$

which is a contradiction. Therefore $\|d(w, w')\| = 0$, i.e., $w = w'$. Thus w is a unique common fixed point of f, g, h and k.

Corollary 2.2. *Let (X, d) be a bicomplex valued metric space and let f and k be* *self-mappings defined on X such that the pair (f, k) be conversely commuting and satisfying*

$$d(fz, fz') \precsim_{i_2} \lambda \max \left\{ \begin{array}{c} \frac{d(kz,kz')+d(kz,fz)}{2}, \frac{d(kz,fz)+d(kz',fz')}{2}, \\ \frac{d(kz,fz')+d(kz',fz)}{2} \end{array} \right\}$$

for all $z, z' \in X$, $\lambda \in (0, 1)$. If the pair (f, k) have a commuting point then (f, k) have a unique common fixed point in X.

Proof. The proof can be established easily by taking $f = g$ and $h = k$ in the proof of Theorem 2.2.

Example 1. Let $X = \{\xi = z_1 + i_2 z_2 \in \mathbb{C}_2 : 0 \leq \|z_1\| < 1, \|z_2\| = 0\}$. Let $d : X \times X \to \mathbb{C}_2$ be the metric, defined by

$$d(\xi_1, \xi_2) = \|x_1 - x_2\| + i_2 \|y_1 - y_2\|$$

for all $\xi_1 = x_1 + i_2 y_1, \xi_2 = x_2 + i_2 y_2, z_1 = x_1 + i_1 y_1, z_2 = x_2 + i_1 y_2$.

Define the maps f, g, h and $k : X \to X$ for $z = \frac{i_1 i_2}{n}, n \in \mathbb{N}$ as follows:

$$f(\xi) = \begin{cases} \frac{i_1 i_2}{n+3} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+4} \text{ if } n \text{ is even} \end{cases}, \quad g(\xi) = \begin{cases} \frac{i_1 i_2}{n+4} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+3} \text{ if } n \text{ is even} \end{cases},$$

$$k(\xi) = \begin{cases} \frac{i_1 i_2}{n+2} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+1} \text{ if } n \text{ is even} \end{cases}, \quad h(\xi) = \begin{cases} \frac{i_1 i_2}{n+1} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+2} \text{ if } n \text{ is even} \end{cases}.$$

The set of all conversely commuting point of the mappings f and k are denoted by $C(f, k) = X - \left\{\frac{i_1 i_2}{n}, n \in \mathbb{N}\right\}$ and all conversely commuting point of the mappings g and h are denoted by $C(g, h) = X - \left\{\frac{i_1 i_2}{n}, n \in \mathbb{N}\right\}$.

Therefore the condition (2.2) are satisfied by f, g, h and k.

3. Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrics under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

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